Supplementary Material

1 Injectivity of the DN Map

In this section, we prove the injectivity of the DN map of the PME. For clarity, we first restate the theorem.

Theorem 1. *Let $T=\infty$ and let $\phi \in C(\bar{S}_\infty)\cap C^\infty(S_\infty)$, $\phi \geq 0$ be of the form $\phi(t,x)=\chi(t)g(x)^{\frac{1}{m}}$ for $(t,x)\in S_\infty$ with $g \in C^\infty(\partial Q)$, $g \geq 0$ and an arbitrary but fixed function $\chi \in C^\infty([0,\infty))$ which is increasing and fulfills $\chi(t)=0$ for $t \leq \frac{1}{2}$ and $\chi(t)=1$ for $t \geq 1$. If $\Lambda^{PME}_{\epsilon^{(i)},\gamma^{(i)}}(\phi)=\Lambda^{PME}_{\epsilon^{(ii)},\gamma^{(ii)}}(\phi)$ for all such ϕ, then $\epsilon^{(i)}=\epsilon^{(ii)}$ and $\gamma^{(i)}=\gamma^{(ii)}$.*

Although the proof is based on [3], this is a generalization of the result from [3] since the equality of the DN maps has to hold for a smaller class of functions ϕ. To prove Theorem 1, we first perform a transformation $v := u^m$ and formulate an equivalent problem for the variable v where we consider the following right hand side and DN map:

$$f(t,x) := \phi^m(t,x) = \chi^m(t)g(x) \qquad \text{and} \qquad \Lambda^v_{\epsilon,\gamma}(f) := \gamma\partial_\nu v|_{[0,\infty)\times\partial Q}. \tag{1}$$

Once this is achieved, we proceed in three main steps: In the first step, we apply a Laplace transformation and proceed to consider the variable $V(h,\cdot)=\int_0^\infty e^{-\frac{t}{h}}v(t,\cdot)\,dt$ with $h>0$. This transformation leads to a time-independent elliptic boundary value problem with a corresponding DN map. The second step consists of a first order asymptotic expansion using the ansatz $V(h,\cdot)=X(h)V_0+R_1(h,\cdot)$ with $X(h):=\int_0^\infty e^{\frac{t}{h}}\chi^m(t)\,dt$ where the variables V_0 and $R_1(h,\cdot)$ themselves fulfill corresponding elliptic boundary value problems. In particular, V_0 gives rise to the definition of a DN map for which much theory already exists, see [11], [9], and from which we can conclude the equality $\gamma^{(i)}=\gamma^{(ii)}$. To prove the equality $\epsilon^{(i)}=\epsilon^{(ii)}$ in the third step, we consider a second order asymptotic expansion of the form $V(h,\cdot)=X(h)V_0+V_1^h+R_2(h,\cdot)$ where again, V_0, V_1^h and $R_2(h,\cdot)$ solve suitable elliptic boundary value problems.

1.1 Elimination of the Time-Dependence: Laplace Transformation

We consider the Laplace transform for $h>0$ and $x\in Q$:

$$V(h,x) := \int_0^\infty \mathrm{e}^{-\frac{t}{h}} v(t,x)\,dt. \tag{2}$$

It holds that $V(h,\cdot)\in H^1(Q)$ for $h>0$. This can be proven using the fact that $v\in L^2(Q_T)$, see [3], and that $\int_{Q_T}|\nabla v(t,x)|^2\,dt\,dx$ grows at most polynomially for $T\to\infty$, see [3, Theorem 1] . Let us now derive a boundary value problem for V. The following derivation deviates from [3] as we derive the strong formulation directly for better readability. Note that the equations below hold in a weak sense, that is, $V(h,\cdot)\in H^1(Q)$ satisfies the corresponding weak formulations. We consider a function $\xi\in C_0^\infty(\mathbb{R})$ such that $\xi(t)=1$ for $|t|\leq 1$ and $\xi(t)=0$ for $|t|\geq 2$. Then we get via partial integration:

$$\nabla\cdot(\gamma(x)\nabla V(h,x)) = \int_0^\infty \mathrm{e}^{-\frac{t}{h}} \underbrace{\nabla\cdot(\gamma(x)\nabla v(t,x))}_{=\epsilon(x)\partial_t v^{\frac{1}{m}}(t,x)}\,dt = \int_0^\infty \mathrm{e}^{-\frac{t}{h}}\epsilon(x)\partial_t v^{\frac{1}{m}}(t,x)\,dt = \int_0^\infty \underbrace{\Big(\lim_{T\to\infty}\xi\Big(\frac{t}{T}\Big)\Big)}_{=1}\mathrm{e}^{-\frac{t}{h}}\epsilon(x)\partial_t v^{\frac{1}{m}}(t,x)\,dt$$

$$=\epsilon(x)\Big(-\int_0^\infty \lim_{T\to\infty}\Big(\underbrace{\frac{1}{T}\xi'\Big(\frac{t}{T}\Big)}_{\substack{\to 0\\ \text{for } T\to\infty}}-\frac{1}{h}\underbrace{\xi\Big(\frac{t}{T}\Big)}_{\substack{\to 1\\ \text{for } T\to\infty}}\Big)\mathrm{e}^{-\frac{t}{h}}v^{\frac{1}{m}}(t,x)\,dt+\Big[\lim_{T\to\infty}\xi\Big(\frac{t}{T}\Big)\underbrace{\mathrm{e}^{-\frac{t}{h}}}_{\substack{=0\\ \text{if } t=\infty}}\underbrace{v^{\frac{1}{m}}(t,x)}_{\substack{=0\\ \text{if } t=0}}\Big]_{t=0}^{\infty}\Big)\frac{1}{h}\epsilon(x)\int_0^\infty \mathrm{e}^{-\frac{t}{h}}v^{\frac{1}{m}}(t,x)\,dt =: \mathcal{N}(h,x).$$

Using the previous assumption (1) on the boundary data, we obtain for the boundary values

$$V(h,x)|_{\partial Q} = \int_0^\infty \mathrm{e}^{-\frac{t}{h}}v(t,x)|_{\partial Q}\,dt = \int_0^\infty \mathrm{e}^{-\frac{t}{h}}f(t,x)\,dt = g(x)\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\chi^m(t)\,dt =: g(x)X(h).$$

In conclusion, $V \geq 0$ is the weak solution of the family of boundary value problems given in the following definition. We also define the corresponding DN map.

Definition 2. Let $v = u^m$ as before and $V(h,\cdot)$ be defined as in Equality (2). Let us consider $g \in C^\infty(\partial Q)$, $g \geq 0$ and $\chi \in C^\infty([0,\infty))$ where χ is an increasing function satisfying $\chi(t) = 0$ for $t \leq \frac{1}{2}$ and $\chi(t) = 1$ for $t \geq 1$. Furthermore, we define for $x \in Q$ and $h > 0$

$$\mathcal{N}(h,x) := \frac{1}{h}\epsilon(x)\int_0^\infty e^{-\frac{t}{h}} v^{\frac{1}{m}}(t,x)\,dt \qquad \text{and} \qquad X(h) := \int_0^\infty e^{-\frac{t}{h}} \chi^m(t)\,dt. \tag{3}$$

Then we define the elliptic boundary problem for $V(h,\cdot)$ and the corresponding DN map as

$$\begin{aligned} \nabla\cdot(\gamma(x)\nabla V(h,x)) &= \mathcal{N}(h,x) \quad \text{for } x \in Q, \\ V(h,x) &= X(h)g(x) \quad \text{for } x \in \partial Q \end{aligned} \qquad \text{and} \qquad \Lambda^h_{\epsilon,\gamma}(g) = X(h)^{-1}\gamma\partial_\nu V(h,\cdot)\big|_{\partial Q} \tag{4}$$

Note that $X(h) \in \mathcal{O}(h)$ as $h \to \infty$ which can be proven by elementary calculations.

Estimates. Our goal is to obtain an estimate for $V(h,\cdot)$. Note that for all estimates in this and the following sections, we collect all arising constants in one placeholder constant C.

Lemma 3. *The right hand side $\mathcal{N}(h,\cdot)$ of the elliptic boundary problem from Definition 2 can be estimated for $h > 0$ as*

$$\|\mathcal{N}(h,\cdot)\|_{L^2(Q)} \leq Ch^{-\frac{1}{m}}\|V(h,\cdot)\|^{\frac{1}{m}}_{H^1(Q)}. \tag{5}$$

Proof. Via Hölder's inequality, we can derive the estimate $0 \leq \mathcal{N}(h,x) \leq h^{-\frac{1}{m}}\epsilon(x)V(h,x)^{\frac{1}{m}}$ for all $x \in Q$. In the following, we deviate from the proof presented by Cârstea et al. [3]. Instead of deriving an estimate for $\mathcal{N}(h,\cdot)$ in the L^∞-norm and then estimating against the $W^{1,p}$-norm for $p > n$, we derive a more intuitive estimate for the L^2-norm. Staying in the setting of $p = 2$ will be useful for the following elliptic estimates. Let $m' \in \mathbb{N}$ such that $\frac{1}{m} + \frac{1}{m'} = 1$. Using the previous estimate for $\mathcal{N}(h,\cdot)$, the boundedness of ϵ in $\bar{Q}$ and Hölder's inequality, we obtain

$$\begin{aligned} \|\mathcal{N}(h,\cdot)\|_{L^2(Q)} &\leq \Big(\int_Q \big(h^{-\frac{1}{m}}\epsilon(x)V(h,x)^{\frac{1}{m}}\big)^2\,dx\Big)^{\frac{1}{2}} \leq Ch^{-\frac{1}{m}}\Big(\int_Q V(h,x)^{\frac{2}{m}}\cdot 1\,dx\Big)^{\frac{1}{2}} \leq Ch^{-\frac{1}{m}}\bigg(\Big(\int_Q \big(V(h,x)^{\frac{2}{m}}\big)^m\,dx\Big)^{\frac{1}{m}}\Big(\int_Q 1^{m'}\,dx\Big)^{\frac{1}{m'}}\bigg)^{\frac{1}{2}} \\ &= Ch^{-\frac{1}{m}}|Q|^{\frac{1}{2m'}}\Big(\int_Q V(h,x)^2\,dx\Big)^{\frac{1}{2m}} \leq Ch^{-\frac{1}{m}}\|V(h,\cdot)\|^{\frac{1}{m}}_{L^2(Q)} \leq Ch^{-\frac{1}{m}}\|V(h,\cdot)\|^{\frac{1}{m}}_{H^1(Q)}. \end{aligned}$$

□

Lemma 4. *Let $V(h,\cdot) \in H^1(Q)$ be the solution to the elliptic boundary value problem from Definition 2. Then the estimate*

$$\|V(h,\cdot)\|_{H^1(Q)} \leq Ch(\|\tilde{g}\|_{H^1(Q)} + h^{-1})$$

holds, where $\tilde{g} := Eg$ is the extension of the boundary data via the trace extension operator $E : H^{\frac{1}{2}}(\partial Q) \to H^1(Q)$.

Proof. Nečas [10, Theorem 5.7] yields the existence of a mapping $E : H^{\frac{1}{2}}(\partial Q) \to H^1(Q)$ such that $(Eg)|_{\partial Q} = g$ for all $g \in H^{\frac{1}{2}}(\partial Q)$. This map corresponds to the right-inverse of the trace operator which is linear and continuous. Therefore, we can define the H^1-extension of the boundary data $gC^\infty(\partial Q) \subset H^{\frac{1}{2}}(\partial Q)$ as $\tilde{g} := Eg$ and we know $\|\tilde{g}\|_{H^1(Q)} \leq \|E\|\|g\|_{H^{\frac{1}{2}}(\partial Q)}$ where $\|\cdot\|$ denotes the operator norm. With this extension of g we now write $V(h,\cdot)$ as

$$V(h,\cdot) = \tilde{V}(h,\cdot) + X(h)\tilde{g}, \tag{6}$$

where $\tilde{V}(h,\cdot)$ fulfills the modified boundary value problem $\nabla\cdot(\gamma(x)\nabla\tilde{V}(h,x)) = \mathcal{N}(h,x) - \nabla\cdot(\gamma(x)X(h)\nabla\tilde{g}(x))$ with boundary condition $\tilde{V}(h,x)|_{\partial Q} = 0$. The corresponding weak formulation is

$$a(\tilde{V}(h,\cdot),\varphi) := \int_Q \gamma(x)\nabla\tilde{V}(h,x)\cdot\nabla\varphi(x)\,dx = -\int_Q \mathcal{N}(h,x)\varphi(x) + \gamma(x)X(h)\nabla\tilde{g}(x)\cdot\nabla\varphi(x)\,dx =: l(\varphi).$$

for all $\varphi \in H^1_0(Q)$. For better readability, we omit the h-dependence of $l(\cdot)$ in our notation. Elementary calculations yield the continuity and coercivity of the bilinear form $a(\cdot,\cdot)$, as well as the continuity of the linear form $l(\cdot)$. The Lax-Milgram Theorem yields the bound

$$\|\tilde{V}(h,\cdot)\|_{H^1(Q)} \leq C(\|\mathcal{N}(h,\cdot)\|_{L^2(Q)} + X(h)\|\nabla\tilde{g}\|_{L^2(Q)}) \leq C(\|\mathcal{N}(h,\cdot)\|_{L^2(Q)} + X(h)\|\tilde{g}\|_{H^1(Q)}).$$

Utilizing the reversed triangle inequality and our previous observation $X(h) \in \mathcal{O}(h)$, we arrive at

$$\|V(h,\cdot)\|_{H^1(Q)} \le C(h\|\tilde{g}\|_{H^1(Q)} + \|\mathcal{N}(h,\cdot)\|_{L^2(Q)}). \tag{7}$$

Our estimate of $\|\mathcal{N}(h,\cdot)\|_{L^2(Q)}$ in Lemma 3 depends on $\|V(h,\cdot)\|_{H^1(Q)}$ itself. To overcome this issue, we instead examine the quantity $\max\{\|V(h,\cdot)\|_{H^1(Q)}, 1\}$ which is an upper bound of $\|V(h,\cdot)\|_{H^1(Q)}$. Ultimatively, we are interested in the asymptotic behaviour for $h \to \infty$. Via the estimate (7), Lemma 3 and basic calculations, we obtain for sufficiently large h that

$$\|V(h,\cdot)\|_{H^1(Q)} \le \max\big(\|V(h,\cdot)\|_{H^1(Q)}, 1\big) \le Ch(\|\tilde{g}\|_{H^1(Q)} + h^{-1}).$$

□

1.2 Equivalence of the Diffusion Coefficients

This subsection focuses on proving the equality $\gamma^{(i)} = \gamma^{(ii)}$ through an asymptotic expansion of $V(h,\cdot)$ that decouples the ϵ- and γ-dependent components. We recover a γ-dependent elliptic boundary value problem with boundary data g for which a corresponding DN map can be defined. Theoretical results for this case already exist, see [11], [9], and can be employed to prove the desired equality. We start with the ansatz

$$V(h,x) = X(h)V_0(x) + R_1(h,x), \tag{8}$$

where V_0 and $R_1(h,\cdot)$ solve the following elliptic boundary value problems.

Definition 5. Let $g \in C^\infty(\partial Q)$ and $\mathcal{N}(h,\cdot)$ as in Definition 2. We define $V_0 \in H^1(Q)$ and $R_1(h,\cdot) \in H^1(Q)$ as the solutions of

$$\begin{aligned} \nabla\cdot(\gamma(x)\nabla V_0(x)) &= 0, \quad \text{for } x \in Q, \\ V_0 &= g, \quad \text{for } x \in \partial Q, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \nabla\cdot(\gamma(x)\nabla R_1(h,x)) &= \mathcal{N}(h,x), \quad \text{for } x \in Q, \\ R_1(h,x) &= 0, \quad \text{for } x \in \partial Q. \end{aligned}$$

The DN map corresponding to the boundary value problem for V_0 is defined as

$$\Lambda_\gamma(g) := \gamma\partial_\nu V_0|_{\partial Q}.$$

The DN map in Definition 5 corresponds to the Calderón DN map $\Lambda_\gamma : H^{\frac{1}{2}}(\partial Q) \to H^{-\frac{1}{2}}(\partial Q)$ which is linear and continuous according to [3], [9] or [11]. Note that both elliptic boundary value problems from Definition 5 are well-posed. It is clear that the ansatz from Equation (8) satisfies the original problem for $V(h,\cdot)$.

Auxiliary results for the first order asymptotic expansion. Based on Definition 5, we examine some properties of the variables V_0 and $R_1(h,\cdot)$. After this, we concentrate on the DN map from Definition 5.

Lemma 6. *Let $V_0 \in H^1(Q)$ and $R_1(h,\cdot) \in H^1(Q)$ be as in Definition 5. Then it holds that $R_1(h,\cdot) \le 0$ and*

$$\|R_1(h,\cdot)\|_{H^1(Q)} \le C(\|\tilde{g}\|_{H^1(Q)}^{\frac{1}{m}} + h^{-\frac{1}{m}}).$$

Furthermore, we get $\|V_0\|_{H^1(Q)} \in \mathcal{O}(1)$ and $\|R_1(h,\cdot)\|_{H^1(Q)} \in \mathcal{O}(1)$ for $h \to \infty$.

Proof. As $\nabla\cdot(\gamma(x)\nabla R_1(h,x)) \ge \mathcal{N}(h,x) \ge 0$, it follows that $R_1(h,\cdot)$ is a subsolution of the elliptic operator $\nabla\cdot\gamma\nabla$. This yields $R_1(h,x) \le \max_{x\in\partial Q} R_1(h,x) = 0$ for all $x \in Q$, see [5, Section 6.4]. Furthermore, we obtain with the Lax-Milgram Theorem, Lemma 3 and Minkowski's inequality that

$$\|R_1(h,\cdot)\|_{H^1(Q)} \le C\|\mathcal{N}(h,\cdot)\|_{L^2(Q)} \le Ch^{-\frac{1}{m}}\|V(h,\cdot)\|_{H^1(Q)}^{\frac{1}{m}} \le Ch^{-\frac{1}{m}}h^{\frac{1}{m}}(\|\tilde{g}\|_{H^1(Q)} + h^{-1})^{\frac{1}{m}} \le C(\|\tilde{g}\|_{H^1(Q)}^{\frac{1}{m}} + h^{-\frac{1}{m}}),$$

which implies $\|R_1(h,\cdot)\|_{H^1(Q)} \in \mathcal{O}(1)$ as $h \to \infty$. Since V_0 is independent of h, it follows that $\|V_0\|_{H^1(Q)} \in \mathcal{O}(1)$ as $h \to \infty$. □

Next, we derive a useful representation of the DN map using the dual pairing of $H^{\frac{1}{2}}(\partial Q)$ and $H^{-\frac{1}{2}}(\partial Q)$.

Lemma 7. *Consider $V_0 \in H^1(Q)$ from Definition 5 with boundary data $g \in C^\infty(\partial Q)$ and the Calderón DN map $\Lambda_\gamma : H^{\frac{1}{2}}(\partial Q) \to H^{-\frac{1}{2}}(\partial Q)$. Let $W_0 \in H^1(Q)$ be a test function with trace $f_0 \in H^{\frac{1}{2}}(\partial Q)$. Then the dual pairing between $f_0 \in H^{\frac{1}{2}}(\partial Q)$ and $\Lambda_\gamma(g) \in H^{-\frac{1}{2}}(\partial Q)$ can be represented as*

$$\langle f_0, \Lambda_\gamma(g)\rangle = \int_Q \gamma\nabla V_0\cdot\nabla W_0\,dx. \tag{9}$$

Proof. We first note that $H^{-\frac{1}{2}}(\partial Q)$ is the dual space of $H^{\frac{1}{2}}(\partial Q)$, see [7, Equation (3.22)] and the dual pairing of $f_0 \in H^{\frac{1}{2}}(\partial Q)$ and $\Lambda_\gamma(g) \in H^{-\frac{1}{2}}(\partial Q)$ can be defined as $\langle f_0, \Lambda_\gamma(g)\rangle := \int_{\partial Q} f_0 \Lambda_\gamma(g)\,ds$. Testing the boundary value problem with W_0, partial integration and identification of the dual pairing yields the assertion. □

With this result, we are able to prove some useful properties of the DN map Λ_γ. The key argument for proving the equivalence of the diffusion coefficients $\gamma^{(i)}$ and $\gamma^{(ii)}$ is to generalize the equality $\Lambda_{\gamma^{(i)}}(g) = \Lambda_{\gamma^{(ii)}}(g)$ for all $g \in C^\infty(\partial Q), g \geq 0$ to all $g \in H^{\frac{1}{2}}(\partial Q)$.

Lemma 8. *If the equality $\Lambda_{\gamma^{(i)}}(g) = \Lambda_{\gamma^{(ii)}}(g)$ holds for all $g \in C^\infty(\partial Q), g \geq 0$, then we can conclude that*

$$\Lambda_{\gamma^{(i)}}(g) = \Lambda_{\gamma^{(ii)}}(g) \quad \textit{for all } g \in H^{\frac{1}{2}}(\partial Q). \tag{10}$$

Proof. The equality $\Lambda_{\gamma^{(i)}}(g) = \Lambda_{\gamma^{(ii)}}(g)$ in fact holds for all $g \in C^\infty(\partial Q)$, not just for non-negative g. For this, we refer to the discussion in the introduction of [3]. To obtain Equation (10), the main idea is to use the density of $C^\infty(\partial Q)$ in $H^{\frac{1}{2}}(\partial Q)$, see [4, Chapter 4.2.1], and the continuity of the DN map. □

Continuation of the main argument. Our first goal is to infer the equality of the Calderón DN maps $\Lambda_{\gamma^{(i)}}(g) = \Lambda_{\gamma^{(ii)}}(g)$ for all $g \in C^\infty(\partial Q), g \geq 0$ from the assumption of Theorem 1 that $\Lambda^{PM}_{\epsilon^{(i)},\gamma^{(i)}}(\phi) = \Lambda^{PM}_{\epsilon^{(ii)},\gamma^{(ii)}}(\phi)$ for all admissible ϕ. As the DN map for the transformed variable v contains the same information as the original DN map, we conclude $\Lambda^{v}_{\epsilon^{(i)},\gamma^{(i)}}(f) = \Lambda^{v}_{\epsilon^{(ii)},\gamma^{(ii)}}(f)$ for all $f \in C(\bar{S}_\infty) \cap C^\infty(S_\infty), f \geq 0$ as in Equation (1). This yields the equality $\Lambda^{h}_{\epsilon^{(i)},\gamma^{(i)}}(g) = \Lambda^{h}_{\epsilon^{(ii)},\gamma^{(ii)}}(g)$ for all $g \in C^\infty(\partial Q), g \geq 0$ via

$$\Lambda^{h}_{\epsilon^{(i)},\gamma^{(i)}}(g) = X(h)^{-1}\int_0^\infty e^{-\frac{t}{h}}\Lambda^{v}_{\epsilon^{(i)},\gamma^{(i)}}\,dt = X(h)^{-1}\int_0^\infty e^{-\frac{t}{h}}\Lambda^{v}_{\epsilon^{(ii)},\gamma^{(ii)}}\,dt = \Lambda^{h}_{\epsilon^{(ii)},\gamma^{(ii)}}(g).$$

From Lemma 6, we know that $\|V_0\|_{H^1(Q)} \in \mathcal{O}(1)$ and $\|R_1(h,\cdot)\|_{H^1(Q)} \in \mathcal{O}(1)$ for $h \to \infty$. It also holds that $X(h)^{-1} \in \mathcal{O}(h^{-1})$ and $\partial_\nu R_1(h,\cdot) \in \mathcal{O}(1)$ as $h \to \infty$. Thus, the ansatz from Equation (8) yields

$$\Lambda^{h}_{\epsilon,\gamma}(g) = X(h)^{-1}\gamma\partial_\nu V(h,\cdot)|_{\partial Q} = \underbrace{\gamma\partial_\nu V_0|_{\partial Q}}_{\in\mathcal{O}(1)} + \underbrace{X(h)^{-1}\gamma\partial_\nu R_1(h,\cdot)|_{\partial Q}}_{\in\mathcal{O}(h^{-1})} = \gamma\partial_\nu V_0(x)|_{\partial Q} + \mathcal{O}(h^{-1}). \tag{11}$$

Therefore, $\Lambda^{h}_{\epsilon,\gamma}$ consists of a term of order $\mathcal{O}(1)$ and a term of order $\mathcal{O}(h^{-1})$. As we already know that $\Lambda^{h}_{\epsilon^{(i)},\gamma^{(i)}}(g) = \Lambda^{h}_{\epsilon^{(ii)},\gamma^{(ii)}}(g)$ for all $g \in C^\infty(\partial Q), g \geq 0$, each term of corresponding order of h must be equal. This implies $\gamma^{(i)}\partial_\nu V_0|_{\partial Q} = \gamma^{(ii)}\partial_\nu V_0|_{\partial Q}$, leading to the conclusion that

$$\Lambda_{\gamma^{(i)}}(g) = \gamma^{(i)}\partial_\nu V_0|_{\partial Q} = \gamma^{(ii)}\partial_\nu V_0|_{\partial Q} = \Lambda_{\gamma^{(ii)}}(g) \quad \text{for all } g \in C^\infty(\partial Q), g \geq 0.$$

In summary, we have derived an elliptic boundary value problem for V_0 with boundary data g whose DN map corresponds to the Calderón DN map satisfying

$$\Lambda_{\gamma^{(i)}}(g) = \Lambda_{\gamma^{(ii)}}(g) \quad \text{for all } g \in C^\infty(\partial Q), g \geq 0. \tag{12}$$

Lemma 8 yields that Equation (12) does not only hold for all such $g \geq 0$ but for all $g \in H^{\frac{1}{2}}(\partial Q)$. We emphasize that the elliptic problem for V_0 from Definition 5 remains well-posed when considering boundary data in $H^{\frac{1}{2}}(\partial Q)$. Finally, we obtain from [11, Theorem 0.1] for $n \geq 3$ and from [9, Theorem 1] for $n = 2$ that

$$\gamma^{(i)} = \gamma^{(ii)}.$$

1.3 Equivalence of the Porosity Coefficients

Lastly, we focus on showing the equality $\epsilon^{(i)} = \epsilon^{(ii)}$ in this subsection. In contrast to the previous subsection, we now consider an expansion of second order. Our ansatz for $V(h,\cdot)$ is

$$V(h,x) = X(h)V_0(x) + V_1^h(x) + R_2(h,x), \tag{13}$$

where V_0 solves the boundary value problem from Definition 5. The boundary value problems for V_1^h and R_2 are given in the following Definition.

Definition 9. Let $\chi \in C^\infty([0,\infty))$ satisfy the conditions stated in Definition 2 and let $\hat{X}(h)$ and $\mathcal{N}_0(h,\cdot)$ be defined as

$$\hat{X}(h) := \frac{1}{h}\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\chi(t)\,dt \quad \text{and} \quad \mathcal{N}_0(h,x) := \epsilon(x)\hat{X}(h)V_0^{\frac{1}{m}}(x). \tag{14}$$

Then we define the boundary value problem for $V_1^h \in H^1(Q)$ and $R_2(h,\cdot) \in H^1(Q)$ as

$$\begin{aligned} \nabla\cdot(\gamma(x)\nabla V_1^h(x)) &= \mathcal{N}_0(h,x), && \text{for } x \in Q, \\ V_1^h(x) &= 0, && \text{for } x \in \partial Q, \end{aligned} \quad \text{and} \quad \begin{aligned} \nabla\cdot(\gamma(x)\nabla R_2(h,x)) &= \mathcal{N}(h,x) - \mathcal{N}_0(h,x), && \text{for } x \in Q, \\ R_2(h,x) &= 0, && \text{for } x \in \partial Q. \end{aligned}$$

We note that $\hat{X}(h) \in \mathcal{O}(1)$ as $h \to \infty$ which can be proven by straightforward computation. The boundary value problems for V_0, V_1^h and $R_2(h,\cdot)$ are well-posed. Similar to the previous subsection, this ansatz clearly satisfies the original problem.

Auxiliary results for the second order asymptotic expansion. We move on to providing some auxiliary results needed for proving the equivalence of the porosity coefficients.

Lemma 10. *Let the assumptions from Theorem 1 be satisfied. Then it holds that* $\mathcal{N}(h,\cdot) - \mathcal{N}_0(h,\cdot) \le 0$.

Proof. Let us consider $\omega_0(t,x) := \chi^m(t)V_0(x)$. Then the function ω_0 solves the problem

$$\epsilon(x)\partial_t\omega_0^{\frac{1}{m}}(t,x) - \nabla\cdot(\gamma(x)\nabla\omega_0(t,x)) = \epsilon(x)V_0^{\frac{1}{m}}(x)\partial_t\chi(t) := \tilde{f}(t,x),$$

with initial condition $\omega_0|_{t=0} = 0$ and boundary condition $\omega_0|_{(0,\infty]\times\partial Q} = v|_{(0,\infty]\times\partial Q}$. Furthermore, it holds $\tilde{f} \ge 0$ since we assume $\partial_t\chi(t) \ge 0$ and $V_0 \ge 0$. Using the maximum principle from Cârstea et al. [3, theorem 1], it follows that $\omega_0 \ge v$ which yields

$$\mathcal{N}(h,x) = \frac{1}{h}\epsilon(x)\int_0^\infty \mathrm{e}^{-\frac{t}{h}}v^{\frac{1}{m}}(t,x)\,dt \le \frac{1}{h}\epsilon(x)\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\omega_0^{\frac{1}{m}}(t,x)\,dt = \frac{1}{h}\epsilon(x)\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\chi(t)V_0^{\frac{1}{m}}(x)\,dt = \mathcal{N}_0(h,x).$$

□

This result can be used to obtain an estimate for $\mathcal{N}(h,\cdot) - \mathcal{N}_0(h,\cdot)$.

Lemma 11. *It holds that* $\|\mathcal{N}(h,\cdot)-\mathcal{N}_0(h,\cdot)\|_{L^2(Q)} \le Ch^{-\frac{1}{m}}\|R_1(h,\cdot)\|_{H^1(Q)}^{\frac{1}{m}}$. *In particular, this estimate implies the asymptotic behaviour* $\|\mathcal{N}(h,\cdot)-\mathcal{N}_0(h,\cdot)\|_{L^2(Q)} \in \mathcal{O}(h^{-\frac{1}{m}})$ *as* $h \to \infty$.

Proof. Let $m' \in \mathbb{N}$ such that $\frac{1}{m} + \frac{1}{m'} = 1$. Using Lemma 10, the fact that $a^{\frac{1}{m}} - b^{\frac{1}{m}} \le (a-b)^{\frac{1}{m}}$ for $m > 1$, $a \ge b \ge 0$ and Hölder's inequality, we obtain

$$\begin{aligned} 0 \le \mathcal{N}_0(h,x) - \mathcal{N}(h,x) &= \frac{\epsilon(x)}{h}\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\left(\omega_0^{\frac{1}{m}}(t,x) - v^{\frac{1}{m}}(t,x)\right)dt \le \frac{\epsilon(x)}{h}\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\left(\omega_0(t,x)-v(t,x)\right)^{\frac{1}{m}}dt \\ &= \frac{\epsilon(x)}{h}\int_0^\infty \mathrm{e}^{-\frac{t}{h}(1-\frac{1}{m})}\left(\mathrm{e}^{-\frac{t}{h}}(\omega_0(t,x)-v(t,x))\right)^{\frac{1}{m}}dt \le \frac{\epsilon(x)}{h}\Big(\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\,dt\Big)^{\frac{1}{m'}}\Big(\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\omega_0(t,x)\,dt - \int_0^\infty \mathrm{e}^{-\frac{t}{h}}v(t,x)\,dt\Big)^{\frac{1}{m}} \\ &= \frac{\epsilon(x)}{h}h^{\frac{1}{m'}}\Big(\int_0^\infty \mathrm{e}^{-\frac{t}{h}}\chi^m(t)V_0(x)\,dt - V(h,x)\Big)^{\frac{1}{m}} = h^{\frac{1}{m'}-1}\epsilon(x)\Big(X(h)V_0(x)-V(h,x)\Big)^{\frac{1}{m}} = h^{-\frac{1}{m}}\epsilon(x)\big(-R_1(h,x)\big)^{\frac{1}{m}}. \end{aligned}$$

In the last step, we insert the ansatz $V(h,x) = X(h)V_0(x) + R_1(h,x)$. Note that $-R_1 \ge 0$ as argued in the proof of Lemma 6. The asserted normwise estimate for $\mathcal{N}_0(h,\cdot) - \mathcal{N}(h,\cdot)$ can be derived similarly to the proof of Lemma 3 for the normwise estimate of $\mathcal{N}(h,\cdot)$, employing Hölder's inequality. With this estimate, it follows that $\|\mathcal{N}_0(h,\cdot)-\mathcal{N}(h,\cdot)\|_{L^2(Q)} \in \mathcal{O}(h^{-\frac{1}{m}})$ for $h \to \infty$ since $\|R_1(h,\cdot)\|_{H^1(Q)} \in \mathcal{O}(1)$ from Lemma 6. □

This finally leads us to the following estimates regarding V_1^h and $R_2(h,\cdot)$.

Lemma 12. *For* V_1^h *and* $R_2(h,\cdot)$ *from Definition 9, the following bounds hold:*

$$\|V_1^h\|_{H^1(Q)} \le C\hat{X}(h)\|V_0\|_{L^2(Q)}^{\frac{1}{m}}, \qquad \text{and} \qquad \|R_2(h,\cdot)\|_{H^1(Q)} \le Ch^{-\frac{1}{m}}\|R_1(h,\cdot)\|_{H^1(Q)}^{\frac{1}{m}}.$$

From these estimates it also follows that $\|V_1^h\|_{H^1(Q)} \in \mathcal{O}(1)$ *and* $\|R_2(h,\cdot)\|_{H^1(Q)} \in \mathcal{O}(h^{-\frac{1}{m}})$ *as* $h \to \infty$.

Proof. As $\mathcal{N}_0(h,\cdot) \geq 0$ and $\mathcal{N}(h,\cdot) - \mathcal{N}_0(h,\cdot) \leq 0$ from Lemma 10, V_1^h is a subsolution and R_2 is a supersolution to the elliptic operator $\nabla\cdot(\gamma\nabla)$. Because of the homogeneous boundary data, Evans [5, Section 6.4] yields $V_1^h \leq 0$ and $R_2(x) \geq 0$, respectively. By applying the Lax-Milgram Theorem and Lemma 11, we obtain

$$\|R_2(h,\cdot)\|_{H^1(Q)} \leq C\|\mathcal{N}(h,\cdot) - \mathcal{N}_0(h,\cdot)\|_{L^2(Q)} \leq Ch^{-\frac{1}{m}}\|R_1(h,\cdot)\|_{H^1(Q)}^{\frac{1}{m}} \quad \text{and} \quad \|V_1^h\|_{H^1(Q)} \leq C\|\mathcal{N}_0(h,\cdot)\|_{L^2(Q)} \leq C\hat{X}(h)\|V_0\|_{L^2(Q)}^{\frac{1}{m}}.$$

Since $\|\mathcal{N}_0(h,\cdot) - \mathcal{N}(h,\cdot)\|_{L^2(Q)} \in \mathcal{O}(h^{-\frac{1}{m}})$, it must therefore also hold that $\|R_2(h,\cdot)\|_{H^1(Q)} \in \mathcal{O}(h^{-\frac{1}{m}})$. Lastly, as $\hat{X}(h) \in \mathcal{O}(1)$ and $\|V_0\|_{L^2(Q)} \in \mathcal{O}(1)$ we may conclude that $\|V_h^1\|_{H^1(Q)} \in \mathcal{O}(1)$. □

We move on to derive a representation of the dual pairing of $H^{\frac{1}{2}}(\partial Q)$ and $H^{-\frac{1}{2}}(\partial Q)$ based on the boundary value problem for V_1^h from Definition 9.

Lemma 13. *Let $V_1^h \in H^1(Q)$ be as in Definition 9. Let W be a smooth, weighted harmonic function, that is $\nabla\cdot(\gamma\nabla W) = 0$, with boundary data $\hat{f} \in C^\infty(\partial Q)$. Then the dual pairing between $\hat{f} \in H^{\frac{1}{2}}(\partial Q)$ and $\gamma\partial_\nu V_1^h|_{\partial Q} \in H^{-\frac{1}{2}}(\partial Q)$ can be computed as*

$$\langle \hat{f}, \gamma\partial_\nu V_1^h|_{\partial Q}\rangle = \hat{X}(h)\int_Q \epsilon(x)V_0^{\frac{1}{m}}(x)W(x)\,dx.$$

Proof. Testing the boundary value problem of V_1^h with the function W, utilizing Definition 9 and partial integration yields

$$\begin{aligned}\int_Q \epsilon(x)\hat{X}(h)V_0^{\frac{1}{m}}(x)W(x)\,dx &= \int_Q \mathcal{N}_0(h,x)W(x)\,dx = -\int_Q \gamma(x)\nabla V_1^h(x)\cdot\nabla W(x)\,dx + \int_{\partial Q}\gamma(x)\partial_\nu V_1^h(x)|_{\partial Q}W(x)|_{\partial Q}ds \\ &= \int_Q \underbrace{\nabla\cdot\big(\gamma(x)\nabla W(x)\big)}_{=0}V_1^h(x)\,dx - \int_{\partial Q}\gamma(x)\underbrace{V_1^h(x)}_{=0}\partial_\nu W(x)ds + \int_{\partial Q}\gamma(x)\partial_\nu V_1^h(x)|_{\partial Q}W(x)|_{\partial Q}ds = \int_{\partial Q}\hat{f}(x)\gamma(x)\partial_\nu V_1^h(x)|_{\partial Q}ds.\end{aligned}$$

The right hand side corresponds to the definition of the desired dual pairing which concludes the proof. □

Lastly, we prove a density result which represents a key argument for proving the equivalence of the porosity coefficients.

Lemma 14. *Let $H, W \in H^1(Q)$ be solutions of $\nabla\cdot(\gamma\nabla H) = 0$ and $\nabla\cdot(\gamma\nabla W) = 0$. Then the span of products of the form γHW is dense in $L^2(Q)$.*

Proof. According to Nachman [9] we can introduce transformed variables $\tilde{H} := \gamma^{\frac{1}{2}}H$ and $\tilde{W} := \gamma^{\frac{1}{2}}W$ such that they solve the transformed problems $-\Delta\tilde{H} + q\tilde{H} = 0$ and $-\Delta\tilde{W} + q\tilde{W} = 0$ with $q := \gamma^{\frac{1}{2}}\Delta(\gamma^{\frac{1}{2}})$. Now we can transform these problems into systems of first order differential equations. For this, we define $U := (\tilde{H}, \nabla\tilde{H})^\top = (U_1, U_2)^\top$ and $U^t := (\nabla\tilde{W}, \tilde{W})^\top = (U_1^t, U_2^t)^\top$ which solve

$$DU + A_1U = 0 \quad \text{and} \quad DU^t - A_2^tU^t = 0 \quad \text{where} \quad D = \begin{pmatrix}\nabla & 0\\ 0 & \nabla\end{pmatrix}, A_1 = \begin{pmatrix}0 & -1\\ q & 0\end{pmatrix}, A_2^t = \begin{pmatrix}0 & q\\ 1 & 0\end{pmatrix}.$$

Together we have U, U^t solve $(D + A_1)U = 0$ and $(-D + A_2^t)U^t = 0$ and q is sufficiently regular as γ is assumed to be sufficiently regular. Then [1, Lemma 3.4] yields that products of the form $U_1U_2^t\tilde{H}\tilde{W} = \gamma HW$ are dense in $L^p(Q), p \geq 1$. □

Continuation of the main argument. Having derived some useful auxiliary results, we are able to concentrate on the key aspects of proving the equality $\epsilon^{(i)} = \epsilon^{(ii)}$. In Subsection 1.2, we have argued that $\Lambda^{PM}_{\gamma^{(i)},\epsilon^{(i)}}(\phi) = \Lambda^{PM}_{\gamma^{(ii)},\epsilon^{(ii)}}(\phi)$ for all $\phi \in C(\bar{S}_\infty) \cap C^\infty(S_\infty), \phi \geq 0$ as in Theorem 1 implies $\Lambda^h_{\gamma^{(i)},\epsilon^{(i)}}(g) = \Lambda^h_{\gamma^{(ii)},\epsilon^{(ii)}}(g)$ for all $g \in C^\infty(\partial Q), g \geq 0$. Similar to Lemma 8, we can argue that the DN map is invariant with respect to shifts by constants, which allows us to drop the assumption $g \geq 0$. Therefore, we conclude that $\Lambda^h_{\gamma^{(i)},\epsilon^{(i)}}(g) = \Lambda^h_{\gamma^{(ii)},\epsilon^{(ii)}}(g)$ for all $g \in C^\infty(\partial Q)$. We may even extend our considerations to $g \in H^{\frac{1}{2}}(\partial Q)$ which can be argued similarly to the proof of Lemma 8. With Equation (13), we obtain

$$\Lambda^h_{\epsilon,\gamma}(g) = \gamma\underbrace{\partial_\nu V_0|_{\partial Q}}_{\in\mathcal{O}(1)} + \underbrace{X(h)^{-1}}_{\in\mathcal{O}(h^{-1})}\gamma\underbrace{\partial_\nu V_1^h|_{\partial Q}}_{\in\mathcal{O}(1)} + \underbrace{X(h)^{-1}}_{\in\mathcal{O}(h^{-1})}\gamma\underbrace{\partial_\nu R_2(h,\cdot)|_{\partial Q}}_{\in\mathcal{O}(h^{-\frac{1}{m}})} = \gamma\partial_\nu V_0|_{\partial Q} + \gamma h^{-1}\partial_\nu V_1^h|_{\partial Q} + \mathcal{O}(h^{-1-\frac{1}{m}}).$$

Note that all summands are of different order of h as $h \to \infty$. Thus, every term of corresponding order in this expansion must be equal for all $g \in H^{\frac{1}{2}}(\partial Q)$, in particular it holds for the pairs $(\epsilon^{(i)}, \gamma^{(i)}), (\epsilon^{(ii)}, \gamma^{(ii)})$ that

$$\gamma^{(i)}\partial_\nu V_1^h|_{\partial Q} = \gamma^{(ii)}\partial_\nu V_1^h|_{\partial Q}. \tag{15}$$

Let W be defined as in Lemma 13. Together with Equation (15), Lemma 13 yields

$$\hat{X}(h)\int_Q \epsilon^{(i)}(x)V_0^{\frac{1}{m}}(x)W(x)\,dx = \langle \gamma^{(i)}\partial_\nu V_1^h|_{\partial Q}, \hat{f}\rangle = \langle \gamma^{(ii)}\partial_\nu V_1^h|_{\partial Q}, \hat{f}\rangle = \hat{X}(h)\int_Q \epsilon^{(ii)}(x)V_0^{\frac{1}{m}}(x)W(x)\,dx,$$

or equivalently

$$\int_Q \left(\epsilon^{(i)} - \epsilon^{(ii)}\right)V_0^{\frac{1}{m}}W\,dx = 0. \tag{16}$$

Note that we only write γ instead of $\gamma^{(i)}$ or $\gamma^{(ii)}$ in the following, having proven the equality of both terms in the previous section. Our ultimate goal is to conclude from this statement that the equality $\epsilon^{(i)} = \epsilon^{(ii)}$ must be fulfilled. So far, we only know that the integral in Equation(16) vanishes for all solutions V_0 of the elliptic boundary value problem from Definition 5 with any boundary data $g \in H^{\frac{1}{2}}(\partial Q)$ and for all weighted harmonic functions with any boundary data $\hat{f} \in C^\infty(\partial Q)$. Via the density of $C^\infty(Q)$ in $H^1(Q)$, this also holds for all weak solutions $W \in H^1(Q)$ of $\nabla\cdot(\gamma\nabla W) = 0$ with arbitrary boundary data $\hat{f} \in H^{\frac{1}{2}}(\partial Q)$. Furthermore, we can show that the integral in Equation (16) also vanishes when only considering the product of two functions without the additional exponent $\frac{1}{m}$ which will be useful in the following. To this end, we start by considering the function V_0 to be of the form $V_0(x) = 1 + sH(x)$ with $s \in \mathbb{R}$ and H being a weak solution of $\nabla\cdot(\gamma\nabla H) = 0$. This is a legitimate assumption since V_0 still fulfills the original problem. Inserting this in Equation (16) yields

$$\int_Q \left(\epsilon^{(i)}(x) - \epsilon^{(ii)}(x)\right)(1 + sH(x))^{\frac{1}{m}}W(x)\,dx = 0.$$

This implies, by considering the derivatives with respect to s on both sides of the equation, that the following statement holds:

$$0 = \frac{d}{ds}\Big|_{s=0}\int_Q \left(\epsilon^{(i)}(x) - \epsilon^{(ii)}(x)\right)(1 + sH(x))^{\frac{1}{m}}W(x)\,dx = \frac{1}{m}\int_Q \left(\epsilon^{(i)}(x) - \epsilon^{(ii)}(x)\right)H(x)W(x)\,dx.$$

Consequently, we obtain that

$$\int_Q \left(\epsilon^{(i)} - \epsilon^{(ii)}\right)HW\,dx = \int_Q \left(\epsilon^{(i)} - \epsilon^{(ii)}\right)\frac{1}{\gamma}(\gamma HW)\,dx. = 0, \tag{17}$$

for all weak solutions H, W of the elliptic boundary value problem with arbitrary, smooth boundary data. Having shown in Lemma 14 that the span of such products γHW is dense in $L^2(Q)$, it follows that $\int_Q \left(\epsilon^{(i)} - \epsilon^{(ii)}\right)\frac{1}{\gamma}\varphi\,dx = 0$ for all $\varphi \in L^2(Q)$. In particular, this holds for all $\varphi \in C_0^2(Q)$. Then the fundamental lemma of the calculus of variations for dimension $n = 2$, see [6, page 22], yields $\left(\epsilon^{(i)} - \epsilon^{(ii)}\right)\frac{1}{\gamma} = 0$ and since we have assumed that $\gamma > 0$ we may finally conclude that $\epsilon^{(i)} = \epsilon^{(ii)}$ almost everywhere. As we have assumed that $\epsilon^{(i)}, \epsilon^{(ii)} \in C^\infty(\bar{Q})$, it follows that

$$\epsilon^{(i)}(x) = \epsilon^{(ii)}(x) \quad \text{for all } x \in Q,$$

which yields the equivalence of the porosity coefficients $\epsilon^{(i)}$ and $\epsilon^{(ii)}$ and thus concludes the proof of Theorem 1. Note that this argument only works for $Q \subset \mathbb{R}^2$. References for extensions to higher dimensions can be found in [3] and [2]. □

2 Proofs Regarding the Numerical Scheme

Lemma 15. *Let the assumptions from Theorem 1 be satisfied, let* $\phi = \chi g^{\frac{1}{m}}$ *be the boundary conditions of the PME and let the DN map* $\Lambda^{PME}_{\epsilon^*,\gamma^*}(\chi g^{\frac{1}{m}})$ *be given. Let* $\lambda(g) := \lim_{h\to\infty}\frac{\int_0^\infty e^{-\frac{t}{h}}\Lambda^{PME}_{\epsilon^*,\gamma^*}(\chi g^{\frac{1}{m}})\,dt}{\int_0^\infty e^{-\frac{t}{h}}\chi^m(t)\,dt}$. *We consider the elliptic equation* $\nabla\cdot(\gamma\nabla v) = 0$ *in* Q *with boundary condition* $v|_{\partial Q} = g$ *and DN map* $\Lambda_\gamma(g) := \gamma\partial_\nu v|_{\partial Q}$. *Then it holds:* $\Lambda_{\gamma^*}(g) = \lambda(g)$.

Proof. In the proof of Theorem 1, we have derived the boundary value problem (4) with corresponding DN map $\Lambda^h_{\epsilon,\gamma}(g) = X(h)^{-1}\gamma\partial_\nu V(h,\cdot)|_{\partial Q}$ with $X(h) := \int_0^\infty e^{-\frac{t}{h}}\chi^m\,dt$. This yields our first intermediate result:

$$\Lambda^h_{\epsilon^*,\gamma^*}(g) = X(h)^{-1}\int_0^\infty e^{-\frac{t}{h}}\gamma^*\partial_\nu v(t,x)|_{S_\infty}\,dt = X(h)^{-1}\int_0^\infty e^{-\frac{t}{h}}\gamma^*\partial_\nu u^m(t,x)|_{S_\infty}\,dt = X(h)^{-1}\int_0^\infty e^{-\frac{t}{h}}\Lambda^{PME}_{\epsilon^*,\gamma^*}(\chi g^{\frac{1}{m}})\,dt.$$

Thus, we are able to compute the DN map $\Lambda^{h}_{\epsilon^*,\gamma^*}$ based on the knowledge of the PME DN map $\Lambda^{PME}_{\epsilon^*,\gamma^*}$. In Subsection 1.2, we have considered the asymptotic expansion $\Lambda^{h}_{\epsilon^*,\gamma^*}(g) = \Lambda_{\gamma^*}(g) + X(h)^{-1}\gamma^*\partial_\nu R_1(h,\cdot)|_{\partial Q}$. From Lemma 6, we know that $\|R_1\|_{H^1(Q)} \in \mathcal{O}(1)$ and it holds $X(h)^{-1} \in \mathcal{O}(h^{-1})$. Therefore, we get our second intermediate result:

$$\lim_{h\to\infty} \Lambda^{h}_{\epsilon^*,\gamma^*}(g) = \lim_{h\to\infty} \left(\Lambda_{\gamma^*}(g) + X(h)^{-1}\gamma^*\partial_\nu R_1|_{\partial Q}\right) = \Lambda_{\gamma^*}(g) + 0 = \Lambda_{\gamma^*}(g).$$

Combining both results and inserting the definition of $X(h)$ yields the assertion. □

Lemma 16. *Let $\{\hat\varphi_j\}_{j=1}^{N_B}$ be the nodal FE basis at the boundary. Let $\lambda^{\mathrm{h}}_{min} > 0$ be the smallest eigenvalue of the mass matrix $M := (\langle\hat\varphi_i,\hat\varphi_j\rangle)_{i,j=1}^{N_B}$. Then we obtain the bound $\|\Lambda_{\gamma,\mathrm{h}} - \Lambda_{\gamma^*,\mathrm{h}}\|_{\mathrm{h}} \le \frac{1}{\sqrt{\lambda^{\mathrm{h}}_{min}}}\sum_{j=1}^{N_B} \|(\Lambda_{\gamma,\mathrm{h}} - \Lambda_{\gamma^*,\mathrm{h}})(\hat\varphi_j)\|_{L^2(\partial Q_{\mathrm{h}})}$.*

Proof. Since the mass matrix M is symmetric and positive definite, there exists an eigenvalue decomposition $M = Q\Lambda Q^\top$ where $Q \in \mathbb{R}^{N_B\times N_B}$ is orthogonal and consists of the normalized eigenvectors of M and $\Lambda \in \mathbb{R}^{N_B\times N_B}$ is a diagonal matrix whose entries correspond to the eigenvalues of the mass matrix M. As a preparation, we also define the vector $\underline{c} := Q^\top \underline{g}$ where $\underline{g}$ is the parameter vector when representing g_{h} in the basis $\{\hat\varphi_j\}_{j=1}^{N_B}$. For boundary data satisfying $\|g_{\mathrm{h}}\|_{L^2(\partial Q_{\mathrm{h}})} = 1$ it holds $\|\underline{c}\|_2^2 \le \frac{1}{\lambda^{\mathrm{h}}_{min}}$. This can be shown via standard computations, using the representation of g_{h} with respect to $\{\hat\varphi_j\}_{j=1}^{N_B}$ and the eigenvalue decomposition of the mass matrix. Since Q and $Q^\top$ are orthonormal matrices, we get the estimate $|g_j| = |(Q\underline{c})_j| \le \|Q_{j,\cdot}\|_2\,\|\underline{c}\|_2 \le \frac{1}{\sqrt{\lambda^{\mathrm{h}}_{min}}}$. The discretized operator norm of the DN maps can then be estimated as

$$\|\Lambda_{\gamma,\mathrm{h}} - \Lambda_{\gamma^*,\mathrm{h}}\|_{\mathrm{h}} \le \sup_{\|g_{\mathrm{h}}\|_{L^2(\partial Q_{\mathrm{h}})}=1} \sum_{j=1}^{N_B} |\underline{g}_j|\,\|\Lambda_{\gamma,\mathrm{h}}(\hat\varphi_j) - \Lambda_{\gamma^*,\mathrm{h}}(\hat\varphi_j)\|_{L^2(\partial Q_{\mathrm{h}})} \le \frac{1}{\sqrt{\lambda^{\mathrm{h}}_{min}}}\sum_{j=1}^{N_B} \|\Lambda_{\gamma,\mathrm{h}}(\hat\varphi_j) - \Lambda_{\gamma^*,\mathrm{h}}(\hat\varphi_j)\|_{L^2(\partial Q_{\mathrm{h}})}.$$

□

Lemma 17. *The forward map $\mathcal{F}_b : \mathcal{D} \subset \mathbb{R}^p \to \mathbb{R}^M, \underline{\gamma} \mapsto \left(\left((\Lambda_{\gamma,\mathrm{h}}(\hat\varphi_1)(x_{\mathrm{m}}))_{\mathrm{m}=1}^{N_B-1}\right)^\top, \ldots, \left((\Lambda_{\gamma,\mathrm{h}}(\hat\varphi_{N_B})(x_{\mathrm{m}}))_{\mathrm{m}=1}^{N_B-1}\right)^\top\right)^\top$ for the Bayesian inversion of the diffusion coefficient is continuous with respect to $\underline{\gamma}$.*

Proof. Let $\mathbf{n}_{\mathrm{m}} := \mathbf{n}(x_{\mathrm{m}}) \in \mathbb{R}^2$, $\Phi_{\mathrm{m}} := (\hat\varphi_1(x_{\mathrm{m}}),\ldots,\hat\varphi_p(x_{\mathrm{m}}))^\top \in \mathbb{R}^p$ and $\Psi_{\mathrm{m}} := (\mathbf{n}_{\mathrm{m}}^\top\nabla\varphi_1(x_{\mathrm{m}}),\ldots,\mathbf{n}_{\mathrm{m}}^\top\nabla\varphi_N(x_{\mathrm{m}}))^\top \in \mathbb{R}^N$ for $\mathrm{m} = 1,\ldots,N_B-1$. Thus, an evaluation of the diffusion coefficient can be represented by $\gamma_{\mathrm{m}} := \gamma(x_{\mathrm{m}}) = \Psi_{\mathrm{m}}^\top\underline{\gamma} \in \mathbb{R}$. Furthermore, the system matrix $A_{\mathrm{h}}(\gamma)$ is invertible for all $\underline{\gamma} \in \mathcal{D}_\gamma$ and continuously depends on $\underline{\gamma}$. As the matrix inversion is a continuous operation for nonsingular matrices according to Meyer [8, Example 6.2.7], this yields the continuity of $A_{\mathrm{h}}^{-1}(\cdot)$ with respect to $\underline{\gamma}$ in $\mathcal{D}_\gamma$. The components of the Bayesian forward map for $\mathrm{m} = 1,\ldots,N_B-1$ and $j = 1,\ldots,N_B$ can thus be written as

$$\Lambda_{\gamma,\mathrm{h}}(\hat\varphi_j)(x_{\mathrm{m}}) = \gamma_m\mathbf{n}_{\mathrm{m}}^\top\nabla V_{0,j}(x_{\mathrm{m}}) = \gamma_{\mathrm{m}}\sum_{i=1}^{N}\underline{v}_i\mathbf{n}_{\mathrm{m}}^\top\nabla\varphi_i(x_{\mathrm{m}}) = \gamma_{\mathrm{m}}\Phi_{\mathrm{m}}^\top\underline{v} = \Psi_{\mathrm{m}}^\top\underline{\gamma}\Phi_{\mathrm{m}}^\top A_{\mathrm{h}}^{-1}(\underline{\gamma})\underline{b}_j.$$

As a concatenation of continuous functions, we may conclude that each component $\Lambda_{\gamma,\mathrm{h}}(\hat\varphi_j)(x_{\mathrm{m}})$ of the forward map is continuous with respect to the parameter vector $\underline{\gamma}$ which yields the assertion. □

Lemma 18. *Let $m \in \mathbb{N}$ be an odd number and let the numerical solution of the PME be defined through a finite difference scheme with step size Δt in time and a linear Lagrange FE scheme in space. Let the resulting system of nonlinear equations be solved numerically using the Newton method with finitely many steps. If Δt is sufficiently small, then the numerical scheme is well-defined and the forward map $\mathcal{F}_e : \mathbb{R}^p \to \mathbb{R}^{\tilde M}, \underline{\epsilon} \mapsto \left(\left((\Lambda^{PME}_{\epsilon,\gamma,\mathrm{h}}(\phi_1)(t_1,x_{\mathrm{m}}))_{\mathrm{m}=1}^{N_B-1}\right)^\top,\ldots,\left((\Lambda^{PME}_{\epsilon,\gamma,\mathrm{h}}(\phi_{N_B})(t_{N_T},x_{\mathrm{m}}))_{\mathrm{m}=1}^{N_B-1}\right)^\top\right)^\top$ for the Bayesian inversion of the porosity coefficient is continuous with respect to $\underline{\epsilon}$.*

Proof. It suffices to argue that the parameter vector $\underline{u}_\epsilon$ continuously depends on $\underline{\epsilon}$. The continuity of the forward map $\mathcal{F}_e$ then follows from the concatenation of continuous operations. In the following, we consider ϵ instead of $\underline{\epsilon}$ for better readability as both quantities can be identified with each other. First, we realize that the entries of $\underline{u}_\epsilon$ corresponding to the boundary nodes are continuous with respect to ϵ as they do not depend on ϵ. The entries of $\underline{u}_\epsilon$ corresponding to the nodes in the interior are determined using the Newton method. Let $\Psi^\top := (\varphi_1,\ldots,\varphi_{N_I})$ where $N_I \in \mathbb{N}$ denotes the number of nodes in the interior of the domain Q. The Newton method is applied to solve the equation

$$R_\epsilon(\underline{u}) := \begin{pmatrix} R_1(\underline{u}) \\ \vdots \\ R_{N_I}(\underline{u})\end{pmatrix} = 0 \quad\text{with}\quad R_i(\underline{u}) = \int_Q \epsilon\frac{\Psi^\top(\underline{u}-\underline{u}_{old})}{\Delta t}\varphi_i + \gamma\nabla(\Psi^\top\underline{u})^m\,dx \quad\text{for } i = 1,\ldots,N_I.$$

Each Newton iteration is given by $\underline{u}^{(\ell+1)} = \underline{u}^{(\ell)} + \underline{z}$ with $\underline{z} = -J_{R_\epsilon}(\underline{u}^{(\ell)})^{-1} R_\epsilon(\underline{u}^{(\ell)})$ where J_{R_ϵ} denotes the Jacobian of R_ϵ with respect to $\underline{u}$. Its entries are given as

$$\frac{\partial}{\partial \underline{u}_j} R_i(\underline{u}) = \int_Q \frac{\epsilon}{\Delta t}\varphi_i\varphi_j\,dx + m\int_Q \gamma(\Psi^T\underline{u})^{m-1}\nabla\varphi_i\cdot\nabla\varphi_j\,dx + m(m-1)\int_Q \gamma(\Psi^\top u)^{m-2}\varphi_j\nabla\varphi_i\cdot\nabla(\Psi^\top\underline{u})\,dx$$

We note that R_ϵ continuously depends on $\underline{u}$ and ϵ and J_{R_ϵ} continuously depends on ϵ. Furthermore, the Jacobian is invertible for sufficiently small Δt. For this, we consider $\alpha \in \mathbb{R}^{N_I} \setminus \{0\}$ and denote $\sum_{i=1}^{N_I} \alpha_i\varphi_i =: v \in H_0^1(Q)\setminus\{0\}$ and $u = \sum_{i=1}^{N_I} \underline{u}_i\varphi_i$. Then

$$\begin{aligned}
\underline{\alpha}^\top J_{R_\epsilon}\underline{\alpha} &= \sum_{i,j=1}^{N_I}\alpha_i\alpha_j\frac{\partial}{\partial\underline{u}_j}R_i(\underline{u}) = \int_Q \frac{\epsilon}{\Delta t}v^2\,dx + m\int_Q \underbrace{\gamma u^{m-1}|\nabla v|^2}_{\geq 0}\,dx + m(m-1)\int_Q \gamma u^{m-2} v\nabla v\cdot\nabla u\,dx\\
&\geq \frac{C_\epsilon}{\Delta t}\|v\|^2_{L^2(Q)} + m(m-1)\int_Q \gamma u^{m-2}v\nabla v\cdot\nabla u\,dx \geq \frac{C_\epsilon}{\Delta t}\|v\|^2_{L^2(Q)} - m(m-1)\underbrace{\left|\int_Q \gamma u^{m-2}v\nabla v\cdot\nabla u\,dx\right|}_{\leq \|\gamma u^{m-2}\nabla u\|_{L^\infty(Q)}\|\nabla v\|_{L^2(Q)}\|v\|_{L^2(Q)}}\\
&\geq \frac{C_\epsilon}{\Delta t}\|v\|^2_{L^2(Q)} - m(m-1)C_{\gamma,u,\nabla u}C_{\text{poincare}}\|v\|^2_{L^2(Q)} = \left(\frac{C_\epsilon}{\Delta t} - m(m-1)C_{\gamma,u,\nabla u}C_{\text{poincare}}\right)\|v\|^2_{L^2(Q)} > 0
\end{aligned}$$

if Δt is sufficiently small such that the factor in the final step is positive. We may conclude that all iterations of the Newton method are well-defined since the system matrix in each step is invertible. Since the matrix inverse is a continuous operation for non-singular matrices, see [8, Example 6.2.7], and the Jacobian is continuous with respect to ϵ, its inverse also continuously depends on ϵ. Mathematical induction yields the continuity of all Newton iterates $\underline{u}_\epsilon^{(\ell)}$ with respect to ϵ for any finite $\ell \in \mathbb{N}$, since each step only involves continuous operations. □

Note that if the initial guess is chosen such that the Newton method converges, the scheme reaches a prescribed accuracy within finitely many steps.

References

[1] Alexander L. Bukhgeim. Recovering a potential from Cauchy data in the two-dimensional case. *Journal of Inverse and Ill-posed Problems*, 16:19–33, 2008. doi: 10.1515/jiip.2008.002.

[2] Cătălin I. Cârstea and Gunther Uhlmann. An inverse problem for the porous medium equation with partial data and a possibly singular absorption term. *SIAM Journal on Mathematical Analysis*, 55:162–185, 2023. doi: 10.1137/21M1465573.

[3] Cătălin I. Cârstea, Tuhin Ghosh, and Gen Nakamura. An inverse boundary value problem for the inhomogeneous porous medium equation. *SIAM Journal on Applied Mathematics*, 85(1):278–293, 2025. doi: 10.1137/23M1623197.

[4] Françoise Demengel and Gilbert Demengel. *Functional Spaces for the Theory of Elliptic Partial Differential Equations*. Universitext. Springer London, 2012. doi: 10.1007/978-1-4471-2807-6.

[5] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 2nd edition, 2010. doi: 10.1090/gsm/019.

[6] Izrail M. Gelfand and Sergei V. Fomin. *Calculus of variations*. Prentice-Hall Mathematics Serie. Prentice-Hall, Inc., 1963.

[7] William C. H. McLean. *Strongly elliptic systems and boundary integral equations*, volume 6 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge university press, 2000.

[8] Carl D. Meyer. *Matrix Analysis and Applied Linear Algebra*, volume 71 of *Other Titles in Applied Mathematics*. Society for Industrial and Applied Mathematics, 1 edition, 2000. doi: 10.1137/1.9780898719512.

[9] Adrian I. Nachman. Global uniqueness for a two-dimensional inverse boundary value problem. *Annals of Mathematics*, 143:71–96, 1996. doi: 10.2307/2118653.

[10] Jindřich Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer Berlin, Heidelberg, 2011. doi: 10.1007/978-3-642-10455-8.

[11] John Sylvester and Gunther Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Annals of Mathematics*, 125(1):153–169, 1987. doi: 10.2307/1971291.