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Damping–induced dispersion and dissipation in simple waveguides

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Abstract: *The goal is to gain a non-dimensional formulation of the complex wavenumber knowing the one-dimensional partial differential equation for wave propagation and the harmonic wave approach. This can be disassembled into its components, namely the imaginary part, showing the decay of a wave, and the real part, showing its spatial propagation. Having an eye on damping in waveguides including internal and external damping, we discuss the resulting consequences, which are frequency-dependent phase and group velocities (dispersion), as well as the energy transport and dissipation. This bears relevance for many domains of physics such as seismic and electromagnetic waves.*

Keywords: waves, dissipation, dispersion, damping, partial differential equations

1 Introduction

Waves are omnipresent. If we take look in our everyday life, we notice that all sounds and noises, that we hear or cause, travel as waves in space. We encounter waves in the transmission of signals in radio communication and they help us to heat food. In water, waves are visible to humans. Light is visible in a certain spectral band corresponding to different wavelengths. Seismic waves let us detect subsurface phenomena and structures or manifest themselves as earthquakes.

The occurrence of waves is linked to the occurrence of dissipation. Dissipation is a process in a dynamic system where internal, bulk kinetic and potential energy is converted into heat and other forms of energy. In a viscoelastic bar, there are two kinds of dissipation. Internal damping is caused by viscoelasticity. External damping is caused by the interaction with the surrounding medium.

Dispersion means that the phase velocity of waves is frequency-dependent. This is particularly visible with light in a prism, where different colors, i.e., wavelengths, are refracted differently. In addition, damping causes dissipation, as we will see soon.

As geotechnical engineers, we draw special attention to waves in porous media relevant for a wide range of applications. During the work on my diploma thesis [7] I considered two different approaches to the analysis of soil dynamical tests, the oscillation-based (modal analysis in Sec. 3) and the wave-based (wave propagation in Sec. 4) approach. In the case of the wave-based approach, it was noticed that apart from the one-dimensional wave equation and the harmonic wave approach, no other equations and correlations could be found in textbooks. This is the motivation for this paper: the formulaic description of the propagation of waves in a dissipative medium. Both, external and internal damping, shall be considered along with their influence on the phase and group velocity. Many of the features we discover in this simple example, we will find again in Biot waves occurring in fluid-saturated porous media. [2, 3]

2 Mathematical model

As a starting point for further explanations, the viscoelastic bar shall be considered, which can be seen as an extension of the well-known telegraph or wave equation. The following dimensional equation describes the viscoelastic bar with linear external damping [5, p. 150], u is the (axial) displacement,

$$\rho A \frac{\partial^2 u}{\partial t^2} + d_{\mathrm{ex}} \rho A \frac{\partial u}{\partial t} - EA \frac{\partial^2 u}{\partial x^2} - d_{\mathrm{in}} EA \frac{\partial^3 u}{\partial x^2 \partial t} = f(t,x). \tag{1}$$

The bar is characterized by its materials mass density ρ, the cross-sectional area A and the Young's modulus E. As it can be seen in Eq. (1), a distinction is made between internal $[d_{\mathrm{in}}] = \mathrm{s}$ and external $[d_{\mathrm{ex}}] = 1/\mathrm{s}$ dissipation due to internal and external damping. Internal damping occurs, for example, due to material models, external damping can result from air friction, among other things.

To be able to generalize the representation, the equation can be formulated non-dimensionally by introducing a reference length L and a reference time $\tau = L/c$ with $c = \sqrt{E/\rho}$, cf. Eq. (2). We refer to c as structural constant to emphasize that it corresponds to the wave speed only in the undamped (nondispersive) case. In mathematical terms it is a linear partial differential equation (PDE) with constant coefficients, specifically a hyperbolic PDE. We note that dissipation mechanisms are often more complex than their description by linear terms; however, linear descriptions turned out to be good approximations in many cases and last but not least "Because linear equations are easy to solve and study, the theory of linear oscillations is the most highly developed area of mechanics.", as V.I. Arnold remarks with slight skepticism [8]. The point of departure for the analysis is the bar equation (1) without forcing ($f \equiv 0$), so its non-dimensional formulation can be written as

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \alpha \frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \beta \frac{\partial^3 \bar{u}}{\partial \bar{x}^2 \partial \bar{t}} = 0, \tag{2}$$

with $\alpha = d_{\mathrm{ex}} \tau$ for external and $\beta = \frac{d_{\mathrm{in}}}{\tau}$ for internal damping.

For the sake of brevity, we will omit the bar denoting non-dimensional quantities and note derivatives with respect to non-dimensional time by dots and with respect to non-dimensional coordinate by primes, so that Eq. (2) reads from now on

$$\ddot{u} + \alpha \dot{u} - u'' - \beta \dot{u}'' = 0. \tag{3}$$

3 Modal analysis

Now that the bar equation in its non-dimensional form is known, we will first show how to solve it in terms of oscillatory modes. On bounded domains with given boundary conditions the first choice is a separation ansatz [5]

$$u(x,t) = X(x)T(t), \tag{4}$$

with the space-dependent function $X(x)$ and the time-dependent function $T(t)$, where x and t are the non-dimensional coordinate and time now. Inserting this ansatz into Eq. (3) leads to

$$X(\ddot{T} + \alpha \dot{T}) - X''\big(T + \beta \dot{T}\big) = 0. \tag{5}$$

After performing the separation and assuming a wisely chosen separation constant k, having gained this wisdom by trial and error, we obtain

$$\frac{X''}{X} = \frac{\ddot{T} + \alpha \dot{T}}{T + \beta \dot{T}} = -k^2. \tag{6}$$

Since $k = \mathrm{const.}$ the partial differential equation (3) resolves into two ordinary differential equations, a spatial and a temporal one,

$$X'' + k^2 X = 0, \tag{7a}$$

$$\ddot{T} + (\alpha + k^2 \beta)\dot{T} + k^2 T = 0. \tag{7b}$$

Both equations correspond to free oscillations of a single-degree-of-freedom-oscillator, (7a) to an undamped and (7b) to a damped one. Their solutions are in the underdamped system (common case) respectively

$$X(x) = C_X \cos(kx) + S_X \sin(kx), \tag{8a}$$

$$T(t) = \hat{T} \sin(\omega t + \varphi_0) e^{-\delta t}, \tag{8b}$$

with the two constants C_X and S_X, the amplitude $\hat{T}$, the angle φ_0, the factor of decay $\delta = \frac{\alpha + k^2 \beta}{2}$ and the damped radian eigenfrequency $\omega = \sqrt{k^2 - \delta^2}$.

Consequently, the final solution is

$$u(t,x) = \big(C \cos(kx) + S \sin(kx)\big) \sin(\omega t + \varphi_0) e^{-\delta t}, \tag{9}$$

where the amplitude phase representation of the time function merged into the constants of the space function ($C = C_X \hat{T}$ and $S = S_X \hat{T}$).

Depending on the boundary conditions there is an infinite number of eigenmodes i, which are given through their spatial k_i and their temporal radian frequency ω_i. The coefficients C_i and S_i are determined by the initial conditions. Known from Hagedorn [5] it can be said, that oscillations are nothing else than standing waves, emerging from waves interfering constructively, while other waves, not contributing to an eigenmode, fade away by destructive interference.

4 Wave propagation

Now that all the fundamentals are known, wave propagation can be considered. Typically, wave propagation is studied on unbounded domains, such as the (semi-)unbounded bar, but also helps gaining insight into oscillations in bounded domains. It is noted that the parameters as well as the dimensionless coordinate and the dimensionless time are real numbers. For the frequency Ω, a more restrictive assumption is made, i.e., it must not be negative:

$$\alpha, \beta \in \mathbb{R}, \tag{10a}$$
$$x, t \in \mathbb{R}, \tag{10b}$$
$$\Omega \in \mathbb{R}^+ \cup \{0\}. \tag{10c}$$

Assuming internal and external damping, the non-dimensional wave equation (3) from Sec. 2 can be used. We adopt the approach of harmonic waves, that works for non-dissipative media and we allow a complex wavenumber κ with the imaginary unit i

$$u = \hat{u} e^{\mathrm{i}(\kappa x - \Omega t)}, \tag{11}$$

with the function u and its amplitude $\hat{u}$, where $u, \hat{u} \in \mathbb{C}$ is valid. Three principal cases can be studied: the undamped, the externally damped and the internally damped case.

4.1 Undamped waveguides

The undamped case is a purely theoretical one, since a complete absence of damping is technically impossible. For completeness as well as for repetition and comparison, the well-known hyperbolic partial differential equation of the undamped case is included

$$\ddot{u} = u''. \tag{12}$$

Eq. 12 follows from Eq. 3 when $\alpha = \beta = 0$. The wave ansatz from Eq. (11) solves Eq. (12) with $\kappa = \Omega$, where Ω denotes a prescribed excitation frequency (radian, non-dimensional). Referring back to dimensional quantities we see wave propagation as $x = ct$ with constant and thus frequency-independent phase velocity $c_\mathrm{p} = c$ and group velocity $c_\mathrm{g} = c$.

4.2 Externally damped waveguides

Now that we have gone through the basics we are going to tackle the real, i.e., damped problem, the raison d'être of this paper. Knowing the wave equation and the harmonic wave ansatz we are investigating the wavenumber κ for the externally damped waveguide at first. With $\beta = 0$, Eq. (3) reduces to

$$\ddot{u} + \alpha \dot{u} - u'' = 0 \tag{13}$$

and on insertion of the wave ansatz (11) gives the dispersion relation

$$-\Omega^2 - \mathrm{i}\Omega\alpha + \kappa^2 = 0. \tag{14}$$

From this the complex and frequency-dependent wavenumber κ is determined by

$$\kappa = \pm\Omega\sqrt{1 + \mathrm{i}\frac{\alpha}{\Omega}}, \tag{15}$$

from which we pick the positive root in the sequel, i.e., the wave traveling in positive x-direction.

Eq. (15) can also be written as

$$\kappa = \Omega\sqrt{a + \mathrm{i}b}, \tag{16}$$

where a is the real part and b the imaginary part of the radicand. The real part κ_R and the imaginary part κ_I of the wavenumber can also be viewed individually. We introduce Z_α as absolute value in the externally damped case. By a transformation of the complex number with

$$r = \sqrt{a^2 + b^2} = Z_\alpha = \sqrt{1 + \left(\frac{\alpha}{\Omega}\right)^2} \tag{17}$$

from the Cartesian coordinates into polar coordinates the following representation appears

$$\kappa = \Omega\sqrt{r}\mathrm{e}^{\mathrm{i}\frac{\varphi}{2}} = \Omega\sqrt{r}\left(\cos\left(\frac{\varphi}{2}\right) + \mathrm{i}\sin\left(\frac{\varphi}{2}\right)\right) = \kappa_\mathrm{R}(\Omega) + \mathrm{i}\kappa_\mathrm{I}(\Omega). \tag{18}$$

Using trigonometric relations in the complex number plane, where the real part a and the imaginary part b of the number are the cathetes to the hypotenuse (or radius) r of the complex number

$$\cos(\varphi) = \frac{a}{r} = \frac{1}{Z_\alpha} \tag{19}$$

and using the half-angle formulas

$$\cos\left(\frac{\varphi}{2}\right) = \sqrt{\frac{1}{2}(1+\cos(\varphi))}, \tag{20a}$$

$$\sin\left(\frac{\varphi}{2}\right) = \sqrt{\frac{1}{2}(1-\cos(\varphi))} \tag{20b}$$

leads to expressions for the real and imaginary part of the complex wavenumber depending only on Ω and on α.

This leads to expressions for the real and imaginary part of the complex wavenumber $\kappa = \kappa_R + i\kappa_I$

$$\kappa_R = \Omega\sqrt{\frac{1}{2}(Z_\alpha + 1)}, \tag{21}$$

$$\kappa_I = \Omega\sqrt{\frac{1}{2}(Z_\alpha - 1)}. \tag{22}$$

4.3 Internally damped waveguides

Using again the harmonic wave ansatz (11), now for the wave equation without external damping, $\alpha = 0$,

$$\ddot{u} - u'' - \beta\dot{u}'' = 0 \tag{23}$$

leads to the dispersion relation

$$-\Omega^2 + \kappa^2 - i\Omega\beta\kappa^2 = 0. \tag{24}$$

From this the complex and frequency-dependent wavenumber κ is determined by

$$\kappa = \Omega\sqrt{\frac{1+i\Omega\beta}{1+(\Omega\beta)^2}}. \tag{25}$$

Exactly as in the externally damped case the same rules can be used to transform the complex number.

Using trigonometric relations in the complex number plane and again using the half-angle formulas (20) leads to expressions for the real and imaginary part of the complex wavenumber depending only on Ω and on β. We introduce Z_β as absolute value in the internally damped case. Setting

$$r = Z_\beta = \sqrt{1+(\Omega\beta)^2} \tag{26}$$

and

$$\cos(\varphi) = \frac{1}{Z_\beta}, \tag{27}$$

we obtain along the lines of the arguments provided for the case of externally damped waveguides

$$\kappa_R = \Omega\sqrt{\frac{1}{2}\frac{Z_\beta+1}{Z_\beta^2}}, \tag{28}$$

$$\kappa_I = \Omega\sqrt{\frac{1}{2}\frac{Z_\beta-1}{Z_\beta^2}} \tag{29}$$

for the real and imaginary part of the complex wavenumber in an internally damped waveguide.

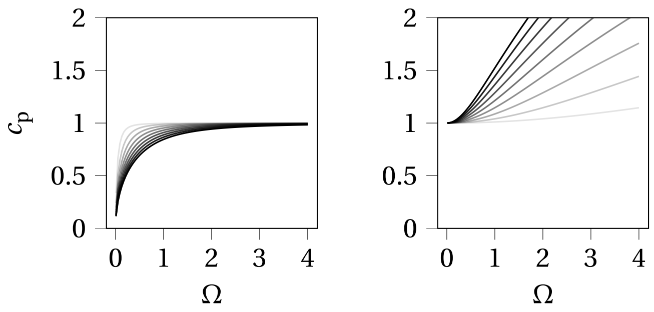

Figure 1 – Phase velocity, externally (left) and internally damped (right), each line corresponds to a damping coefficient, α or β, in the range from 0 (transparent) to 1.5 (opaque).

5 Discussion

If we take now a closer look at wave ansatz (11), we can see the roles of the real and imaginary part of the complex wavenumber. For the chosen sign in positive x-direction (opposite direction for negative sign) we get the following representation

$$e^{i\left((\kappa_R+i\kappa_I)x-\Omega t\right)} = e^{-\kappa_I x}e^{i(\kappa_R x-\Omega t)}, \tag{30}$$

where κ_I leads to decay and κ_R to propagation in space. We confirm that $\kappa_R, \kappa_I \in \mathbb{R}^+$, since the value of the complex number $Z \geq 1$ by the assumptions (10) made on α, β and Ω.

5.1 Dispersion

Dispersion is a common phenomenon when dealing with waves. In this context, the phase and the group velocity have to be distinguished. In a dispersive medium the phase velocity, defined by

$$c_p = c_p(\Omega) = \frac{\Omega}{\kappa_R} \tag{31}$$

depends on the frequency. The specific expressions for externally and internally damped waves are

$$c_p^{ex} = \sqrt{\frac{2}{Z_\alpha+1}}, \tag{32}$$

$$c_p^{in} = \sqrt{\frac{2Z_\beta^2}{Z_\beta+1}} \tag{33}$$

by using equations (21) and (28). At this point we have to remember that $\Omega = \kappa_{un}$, where κ_{un} is the undamped wavenumber, leads to $c_p^{un} = 1$.

For the group velocity c_g in non-dissipative media there is an undisputed definition [1], whereas for dissipative media there exist different definitions [4]. We focus on the propagation, i.e., κ_R. In other words, we look

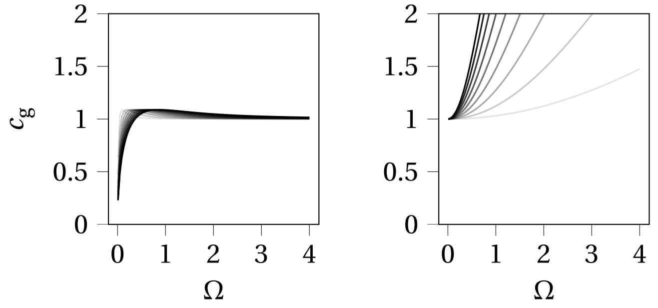

Figure 2 – Group velocity, externally (left) and internally damped (right), each line corresponds to a damping coefficient, α or β, in the range from 0 (transparent) to 1.5 (opaque).

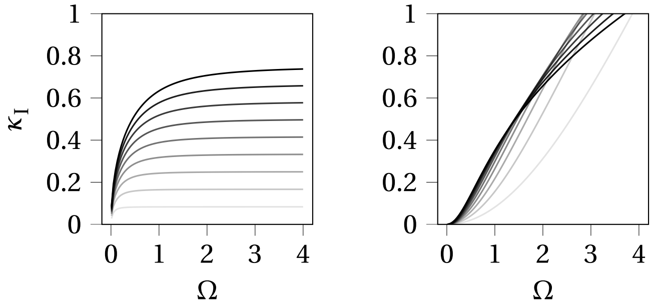

Figure 3 – Imaginary part of wavenumber (decay), externally (left) and internally damped (right), each line corresponds to a damping coefficient, α or β, in the range from 0 (transparent) to 1.5 (opaque).

at the waves through imaginary glasses that compensate the decay by κ_I so that we can apply the common definition of group velocity in non-dissipative media. For the evaluation we use the inverse function rule

$$c_\mathrm{g} = \frac{\partial \Omega}{\partial \kappa_\mathrm{R}} = \left(\frac{\partial \kappa_\mathrm{R}}{\partial \Omega}\right)^{-1} \tag{34}$$

for externally and internally damped waves

$$c_\mathrm{g}^\mathrm{ex} = \frac{2\sqrt{2}\, Z_\alpha \sqrt{Z_\alpha + 1}}{Z_\alpha^2 + 2 Z_\alpha + 1}, \tag{35}$$

$$c_\mathrm{g}^\mathrm{in} = \frac{2\sqrt{2}\, Z_\beta^3 \sqrt{Z_\beta + 1}}{Z_\beta^3 + Z_\beta + 2}. \tag{36}$$

Both relations as seen in Eqs. (32), (33), (35) and (36) give a velocity of one in the undamped case, consistent with what has been discussed above. This applies to both c_p and c_g as shown in Figs. 1 and 2. For external damping the asymptote for decreasing damping coefficients approaches one. So the higher the damping, the more extended the relevant frequency range is. The group velocity in externally damped waveguides shows non-monotonic behaviour with a maximum. A standard analyis of Eq. (35) reveals that this maximum is located at $\arg\max\big(c_\mathrm{g}^\mathrm{ex}(\Omega)\big) = \frac{\alpha}{\sqrt{3}}$ and has the value $\max\big(c_\mathrm{g}^\mathrm{ex}(\Omega)\big) = \frac{4}{9}\sqrt{6}$ for a positive frequency Ω. For internal damping we find that the higher the damping coefficient the higher the group or phase velocities are for a given frequency. This can be related to the apparent stiffening of viscoelastic materials at increasing strain rates. We further observe that the increase with frequency gets more pronounced. In other words, the more damping the stronger the dispersion.

5.2 Dissipation

Since the total energy of a wave is proportional to its squared amplitude, we find the same decay with $e^{-\kappa_\mathrm{I} x}$ along x for both externally and internally damped cases. As shown in Fig. 3 the imaginary part of our complex wavenumber is, in the externally damped case, asymptotic for each damping coefficient. The higher the damping coefficient, the higher the decay of the wave. However, as apparent from Fig. 3, there is a difference in the asymptotic behavior for internal and external damping. For external damping, κ_I approaches a constant value at higher frequencies.

The high-frequency limit can be taken by

$$\lim_{\Omega\to\infty} \kappa_\mathrm{I}(\Omega) = \lim_{Z_\alpha \to 1} \sqrt{\frac{1}{2}\frac{\alpha^2}{Z_\alpha + 1}} = \frac{\alpha}{2}. \tag{37}$$

Internal damping leads to monotonic increase of κ_I with Ω. In the internally damped case, the imaginary parts of the wavenumber are all close together for the different damping coefficients. For lower frequencies, the imaginary part increases with a higher damping coefficient. But with increasing Ω there is a reversal point where higher damping coefficients lead to lower imaginary parts which means a lower decay. We assume that this is related to the apparent viscoelastic stiffening and the resulting different amplitudes, translating into different dissipated work.

5.3 Relation between oscillations and waves

The position of the spatial radian frequency k and the real part of the wavenumber κ_R in solutions (9) and (30), respectively, suggests a relation between them. Note that the discrete values of k are determined by the boundary conditions. Similarly, knowing the wave velocity, we may want to compare the decay constant δ with the product $\kappa_\mathrm{I} c_\mathrm{p}$. Indeed, in the undamped case we have $k = \kappa = \kappa_\mathrm{R}$ and trivially $\delta = \kappa_\mathrm{I} c_\mathrm{p} = 0$. However, this should not mislead us to the incorrect conclusion k and κ_R, as well as δ and $\kappa_I c_p$ were generally equal. Com-

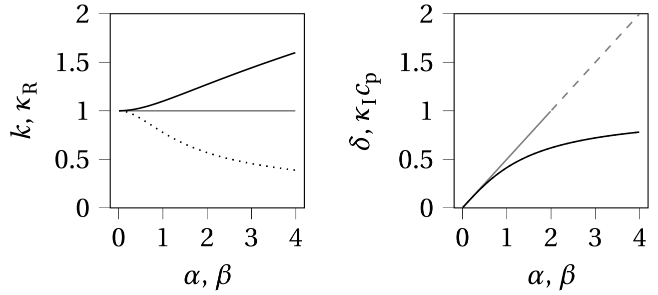

Figure 4 – Wavelength characteristics (left figure) and decay characteristics (right figure) depending on external (black solid line) and internal (black dotted line) damping. The gray line indicates the constant spatial frequency k in the left figure and the decay constant δ in the right figure. Note that the decay characteristics (right figure) for internal and external damping coincide for chosen $k = 1$. Also note that the dashed line for $\delta > 1$ does not correspond to an oscillatory solution anymore, as it is overdamped.

paring them in Fig. 4 for $k = 1$ reveals that they approximately go together for slightly damped cases ($\delta \ll 1$), but generally they differ. The decisive difference is, that in the modal analysis the damping enters as a function of time $e^{-\delta t}$ whereas in the wave ansatz it enters as a spatial function $e^{-\kappa_I x}$. If we want to describe the simple case of a base mode oscillation with the wave ansatz, then we would have to represent the initial conditions as a series, e.g. $u_0(x) = \hat{u}\sin(kx) = \sum_n \hat{u}_n e^{i\kappa_n x}$. Having made some first steps in the supplementary material, we leave this comparison, including interpretations of κ in terms of oscillations in the overdamped regime, for future work.

6 Summary

Waves and oscillations are two manifestations of the same phenomenon. Typically modal analysis is used for bounded domains, whereas the wave ansatz is used for unbounded domains or the transient phase until reflected waves start interfering.

Here we showed a qualitative comparison between externally and internally damped waveguides in terms of different velocities and similar decay. Their dispersion and dissipation characteristics are qualitatively different, namely the dependence of velocity and decay on frequency.

For dispersion, the phase and the group velocity, which are both depending on the frequency, have to be distinguished. The phase and the group velocity are one in the undamped case. For external damping the asymptote for decreasing damping coefficients approaches one, whereby a non-monotonic behavior can be observed for the group velocity. In the internally damped case, the velocities increase with increasing damping coefficients. For dissipation, the imaginary part of the complex wavenumber is asymptotic for each damping coefficient in the externally damped case. The higher the damping coefficient, the higher the decay of the wave. Internal damping leads to a monotonic increase of the imaginary part, i.e., the decay.

The qualitative difference can be used to guide modelling efforts based on experimentally observed scaling behaviour, in particular the choice of damping models for real systems with complex dissipation mechanisms. As suggestions for future work, an explanation in terms of physics of the maximal group velocity for externally damped waveguides and of the crossing points of the decay (imaginary part of wavenumber) for internally damped waveguides would be interesting, as well as a more detailed conversion from oscillation to wave parameters.

Code Availability: Jupyter-notebooks (Python) for the simulation, are available as supplementary material and can be obtained under the

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This format has been used for teaching at TU Bergakademie Freiberg [6].

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