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An experimental numerics approach to the terrestrial brachistochrone

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Abstract: *We revisit the classical and solved problem of the terrestrial brachistochrone, the fastest path between two points in earth's gravitation field, by an approach we refer to as* experimental numerics. *By this term we mean arriving at a qualified guess by deliberately taking inspiration from numerical results that are easily available. Since in many cases verification is easier than derivation, this approach may have some educational merits. Current software tools such as* Jupyter Notebooks *blend coding with documentation and allow leveraging this approach to enable new ways in modern teaching. The intended audience are graduate students with prior knowledge of multivariate calculus, ordinary differential equations (ODEs), calculus of variations and classical physics, particularly mechanics.*

Keywords: numerics, brachistochrone, variational calculus, geophysics, teaching

1 Introduction

How to solve problems? And how to learn it? We try, we wander, we may have some intuition about the solution. Based on this insight or intuition we are willing to try hard to solve certain integrals or differential equations and further difficulties piling up until they once again reduce and *"it clicks"*. Seeing only the streamlined solution or even end result may hide all the effort required in getting there, so that reproducing the solution may appear deceivingly easy. However, if you are faced with having to solve the problem on your own without the proper path being indicated to you, you may not even have an idea where to start. Here, we discuss a way that may be more intuitive for students, or generally of interest when tackling a new problem with a yet unknown solution. Along this way we depart from simpler related problems and continue by exploiting numerical solutions to *"see what happens"* until, finally, we have gained enough insight to anticipate a possible solution. If we are lucky, we can verify the guessed solution as being correct and find motivation to look for alternative routes of solving the problem. This entire procedure can be documented conveniently in interactive notebooks, such as Jupyter[1], and used in a computer-based or demo-centred class. As a precaution, we point out that numerics extend but do not entirely replace analytical methods, especially, but by no means exclusively, in teaching. Let's follow this way from start to end with the example of the terrestrial brachistochrone.

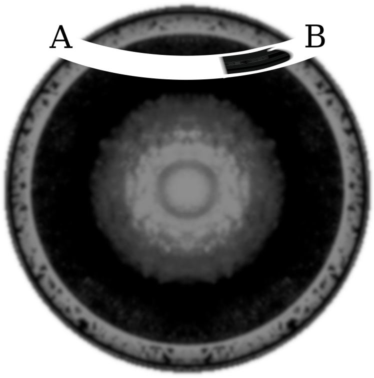

Figure 1 – Gravity train tunnel through the earth

2 The Departure – Stating the Problem

As the translation of *brachistochrone* suggests, we are looking for a path of shortest time, in common parlance the fastest path. The path of a point mass in a force field between two points, for example. The force field still needs to be specified; with the attribute *terrestrial* we refer to a radial field which approximates earth's gravitation well. Deliberately leaving some geotechnical problems to the practitioners, we may imagine a tunnel for a gravity train [3]. At some point A the train slides down into the ground and—under the action of gravity—first accelerates while going deeper into the earth and then decelerates on its path back towards the surface, until it resurfaces at another point B as illustrated in figure 1. Having learned how to describe extrema mathematically we go even further and look for the optimal tunnel trajectory to get from one surface point to the other in the shortest amount of time[2]. In order to arrive at the solution, we need to solve several *interesting integrals* [6]. Of course, some of you may be intrinsically motivated to take this challenge, but others, particularly practically motivated engineering students, might prefer to anticipate the solution first before committing to any mathematical adventures.

For students with a geophysical background we note that the propagation of seismic waves is physically something completely different—but by invoking some common assumptions such as linear elasticity and depth-dependent stiffening, we end up with a mathematically equivalent problem.

[1] `https://jupyter.org/`

[2] The solution is already known [7]—to solve it by yourself nevertheless requires some endurance.

3 The First Steps – Writing Equations

We start from an analytical functional expressing the quantity we wish to minimize; in our case the time required to move from point A to point B. Since we accelerate fastest when going straight towards the earth's center—in other words we would have a less exciting trip surfing equipotential surfaces[3]—we reduce the originally three-dimensional problem to two dimensions on an orthodrome slice (figure 2) through an idealized spherical (no, not flat) earth. Working with polar coordinates r and θ for convenience, the travel time is given by the functional of velocity $v(r)$

$$T = \int_{\mathrm{A}}^{\mathrm{B}} \frac{\mathrm{d}s}{v(r)}. \tag{1}$$

The arc-length s can be expressed geometrically. Likewise, we know the velocity due to the distributed mass (or, alternatively, learn about it in Appendix A.1)

$$\mathrm{d}s = \sqrt{(\mathrm{d}r)^2 + (r\,\mathrm{d}\theta)^2}, \tag{2}$$

$$v(r) = C\sqrt{R^2 - r^2} \quad \text{with} \quad C = \sqrt{g/R}, \tag{3}$$

where g denotes gravitational acceleration and R earth radius. Inserting Eqs. (2) and (3) in (1) specifies our task to find the optimal path $r(\theta)$ that minimizes travel time

$$T = \frac{1}{C}\int_{\theta_{\mathrm{A}}}^{\theta_{\mathrm{B}}} \underbrace{\sqrt{\frac{(\mathrm{d}r/\mathrm{d}\theta)^2 + r^2}{R^2 - r^2}}}_{L}\,\mathrm{d}\theta.$$

This is a typical problem to be solved with the calculus of variations. The integrand, also referred to as *Lagrangian* for this kind of problems, is of the structure $L(r, r')$. Consequently, Noether's theorem [4] applies and we know that there is an associated conserved quantity

$$H = r'\frac{\partial L}{\partial r'} - L$$

along the optimal curves. For those not familiar with Noether's theorem, we explain it for our specific case in Appendix A.2. For reference, we refer to the lowest point r_0, or in other words the point of minimum distance to earth's center. Since it is a minimum, we have $r' = 0$ (necessary condition). So from $H(r, r') = H(r_0, 0)$ we obtain after some simplification

$$r' = \frac{Rr}{r_0}\sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}. \tag{4}$$

This equation determines the optimal path $r(\theta)$.

[3] By symmetry we assume all points of the same radius sharing the same potential.

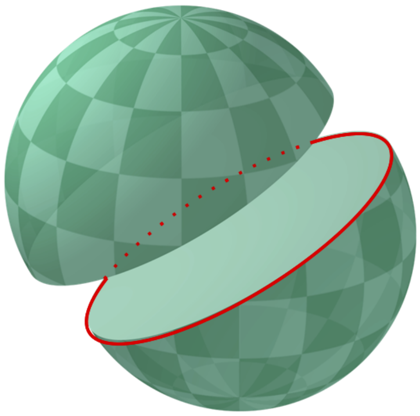

Figure 2 – An orthodrome of a sphere is the intersection of the sphere and a plane that passes through the center point of the sphere [9]

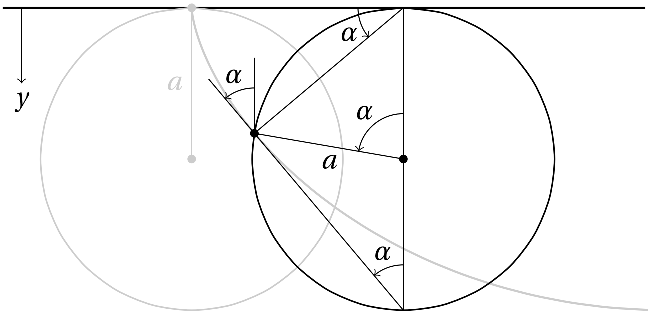

Figure 3 – The cycloid curve solves the classical brachistochrone problem

4 En Route – Getting Ideas

Before we continue with our solution we take inspiration from the landscape surrounding us to remember the classical brachistochrone. After that we are going to solve the ordinary differential equation (4) numerically.

4.1 Known Solution of a Simpler Similar Problem

This section rather recaps the classical brachistochrone, than explains it. Since it is part of almost any course or textbook on calculus of variations, we refer to the literature [4] for a detailed introduction.

This problem assumes a velocity, that depends on the

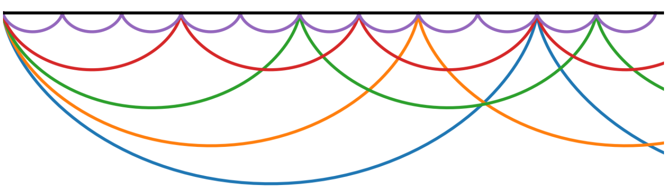

Figure 4 – Cycloids of different radii

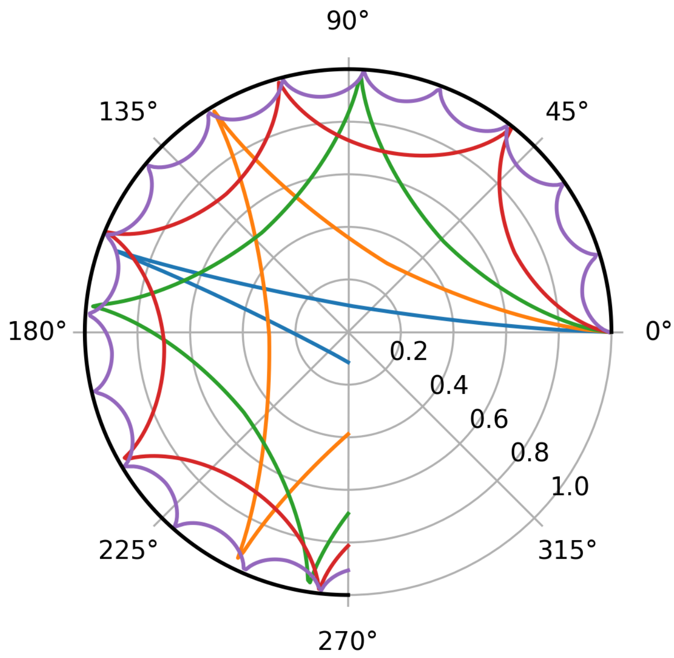

Figure 5 – Numerical solutions for the path of shortest time through earth for different values of the minimal radius $\frac{r_0}{R} \in \{0.1, 0.3, \dots, 0.9\}$ on the domain $\theta \in [0, \frac{3}{2}\pi]$

vertical coordinate y only (conservation of energy)

$$v = \sqrt{2gy}.$$

The travel time to be minimized reads

$$T = \int_{s_A}^{s_B} \frac{\mathrm{d}s}{v} = \int_{x_A}^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,\mathrm{d}x.$$

Applying Noether's theorem, as we did before, results in

$$H = -\frac{1}{\sqrt{2gy}} \frac{1}{\sqrt{1+y'^2}} = \text{const.}$$

Finally, expressing the slope via its angle of inclination $y'(x) = \frac{\cos\alpha}{\sin\alpha}$ and setting forward-looking $H = \frac{-1}{\sqrt{2g}\sqrt{2a}}$ reveals a cycloid

$$\sqrt{y} = \sqrt{2a}\sin\alpha,$$

i.e., the motion of a point on a circle rolling along a straight line, as shown in figure 3. The radius a is determined by the locations of start- and endpoint; some example paths are shown in figure 4.

4.2 Experimental Numerics

With this initial inspiration from the break, we now visualize the solutions for the terrestrial brachistochrone. Although we may have no idea, yet, about an analytical solution, integrating the ODE (4) is a piece of cake when you have numerical methods at your disposal. Without loss of generality, we consider r as a relative radius and set earth radius to unity[4] $R = 1$. Then, the problem is fully parameterized by the radius to the deepest point r_0. So, we choose a set of plausible r_0 values that fit comfortably inside the earth, say $0 < r_0 < 1$, and determine a corresponding range for the independent variable $\theta = [0, \pi]$. Of course, this range could also be found by trial and error. But one may reason by the limit case of passing through the earth's center, $r_0 = 0$, on a straight trajectory connecting two points on opposite sides of the world, that all curves for $r_0 > 0$ should return to the surface for some $\theta < \pi$. Another issue is related to the right-hand side of equation (4), which approaches zero or infinity as $r \to r_0$ or $r \to R$, respectively. In a course on numerical methods for ODEs, this would be the right moment to reiterate the necessity of Lipschitz-continuity for uniqueness of a solution [1]. However, our engineering approach starts numerical integration slightly above r_0 and stops shortly before reaching R, thus avoiding the critical points.

Following the learning-by-playing paradigm, we encourage you to inspect the details on the implementation and to run your own numerical experiments with the supplementary Jupyter script, which is linked in the Code Availability section on page 33.

As we are eager for inspiration, the important moment comes now, when looking at the solutions in figure 5. We observe that the curves flatten the deeper we pass by earth's centre. The more shallow curves remind us of the classical brachistochrone we just revisited. This is, of course, entirely plausible as near-surface trajectories match the assumption of the classical problem quite well: constant gravity and earth's curvature looking locally essentially like a straight line. In other words, we may also think of the classical brachistochrone as limit case of a uniform gravitational field on a flat earth (disc with zero curvature). In addition, we observe the optimal paths starting and ending orthogonal to earth's surface, which may indicate ideal rolling (instantaneous center of rotation). Since the classical brachistochrone is solved by a circle rolling on a straight line (cycloid), the evidence we now have may lead us to the bold conclusion that the terrestrial brachistochrone is solved by a circle rolling inside a bigger circle as shown in figure 6, a curve called the hypocycloid.

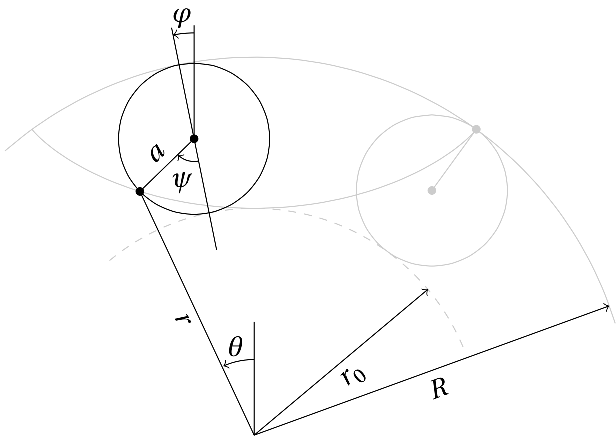

Figure 6 – The hypocycloid curve solves the terrestrial brachistochrone problem

5 The Finale – Verifying the Guess

We are now highly motivated to check whether our guess of the hypocycloid satisfies the optimality criterion, equation (4). To do so, we read the Law of Cosines from figure 6.

$$r^2 = (R-a)^2 + a^2 - 2(R-a)a\cos\psi \tag{5}$$

The angles φ and ψ are related by the rolling condition

$$R\varphi = a\psi$$

and the relation with the angle θ becomes visible via the height of the triangle with the sides a, r and the connection of the circles' centre points

$$\tan(\theta - \varphi) = \frac{a\sin\psi}{R - a(1+\cos\psi)}. \tag{6}$$

Fully resolving $r(\theta)$ seems hard. So instead, we choose ψ as an intermediate variable and apply the chain rule

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}\theta}. \tag{7}$$

It is tempting to differentiate the Law of Cosines (5) to obtain $\frac{\mathrm{d}r}{\mathrm{d}\psi}$, but this would leave us with $\sin\psi$. We did this, but had trouble to continue. However, the tangent half-angle formula

$$\tan\frac{\psi}{2} = \sqrt{\frac{1-\cos\psi}{1+\cos\psi}}$$

contains only $\cos\psi$ which we have as function of the radial coordinate, equation (5). Differentiation then gives

$$\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{2r\sqrt{\frac{r^2-r_0^2}{R^2-r^2}}}{r^2-r_0^2}. \tag{8}$$

[4] We could also formally non-dimensionalize the problem.

To get the remaining second term of the chain rule (7), we differentiate relation (6) with respect to the angle ψ and then substitute $\cos\psi$ and $\sin\psi = \sqrt{1-\cos^2\psi}$ by the Law of Cosines (5)

$$\frac{\mathrm{d}\theta}{\mathrm{d}\psi} = \frac{r_0\Delta_1\left(\Delta_2\sin^2\psi + 2r_0\Sigma_1\left(1+\cos\psi\right)\right)}{2R\left(\Sigma_1-\Delta_1\cos\psi\right)\left(\Sigma_2-\Delta_2\cos\psi\right)}$$

with

$$\begin{aligned}\Delta_1 &= R - r_0\\ \Delta_2 &= R^2 - r_0^2\\ \Sigma_1 &= R + r_0\\ \Sigma_2 &= R^2 + r_0^2\end{aligned}$$

and on further simplification

$$\frac{\mathrm{d}\theta}{\mathrm{d}\psi} = \frac{Rr_0}{2r^2} - \frac{r_0}{2R}. \tag{9}$$

Inserting both factors, the reciprocals of (8) and (9), in the chain rule (7) we see that the hypocycloid indeed satisfies equation (4) and our guess is correct.

6 Summary

Based on the example of the terrestrial brachistochrone, we have demonstrated a prototypical way of problem solving in engineering mathematics and how numerical experiments may help us to come up with a qualified guess. The verification of that guess is not trivial, but at least we know what we are looking for, and this often seems easier than the direct approach. In addition, this is how many engineers have come to work: first running powerful simulation tools, recognizing patterns and then reducing the problem to extract its essentials. The Jupyter ecosystem we use for teaching[5], conveniently integrates theory with applied or even interactive numerics and allows the students to experiment with it (learning by playing). We believe this a promising way of using and teaching both the subject as well as the use of modern software tools.

Code Availability:
The source code of the implementations used to compute the presented results is available as supplementary material and can be obtained under the

doi: 10.14464/gammas.v4i1.512.

[5]For example:
`https://github.com/nagelt/Numerical_Methods_Introduction`
`https://github.com/nagelt/Teaching_Scripts`
`https://github.com/nagelt/soil_dynamics`

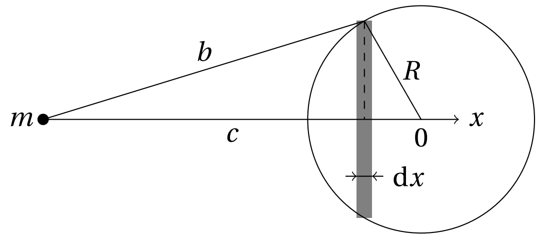

Figure 7 – Two-dimensional projection of a hollow sphere and a circle of latitude on this sphere

The code, as we use it for the course *Soil Dynamics* (in German) at TU Bergakademie Freiberg, is also available at GitHub and also at a ready-to-use web interface.

Acknowledgements: We are grateful to Matthew Van Koevering for his inspiring video *Terrestrial Brachistochrone (Gravity Train)* on YouTube.

A Appendix

A.1 Gravitational Force of a Spherically Distributed Mass

Newton's shell theorem is the bridge between Newton's law of universal gravitation

$$F = G\frac{m_1 m_2}{c^2}, \tag{10}$$

where G denotes the gravitational constant and c the distance between point masses m_1 and m_2, and formula (3). The shell theorem states [2]: *If an object lies outside a thin, uniform shell of mass, then the vector sum of all the gravitational forces exerted by all the parts of the shell is the same as if all the shell's mass was concentrated at its center. If the object lies inside the shell, then all the gravitational forces cancel out exactly.* Finally, it is straightforward to compose a solid earth of an infinite number of infinitesimally thin shells and omitted for sake of brevity.

The standard way would be the three-dimensional integration over the distributed mass, however in the illustrative spirit of this paper, we are happy to have found the idea to proceed via the hat box theorem [5], which we are going to summarize here.

Let us consider the total gravitational force by equation (10) exerted by a distributed mass on a point mass. Figure 7 illustrates how we slice the hollow sphere into circles of latitude (spherical segment). All force components orthogonal to the line, connecting the sphere center and the point mass m, cancel (opposite force component on the opposite side of the circle). What remains is the component parallel to this line

$$\mathrm{d}F_c = G\frac{m\,\mathrm{d}M}{b^2}\frac{c+x}{b}. \tag{11}$$

Note the sign of x determined by the coordinate axis. The hat box theorem [8] states that all spherical segments with the same thickness have the same area. This predestines $\mathrm{d}x$ in integration over the total mass, since each spherical segment of thickness $\mathrm{d}x$ has the same mass

$$\mathrm{d}M = K\,\mathrm{d}x.$$

The constant K is determined by the total mass

$$M = \int_{-R}^{R} K\,\mathrm{d}x = 2KR,$$

where we assumed a constant density. Since we made our choice for x as independent variable, we need to find $b(x)$. From recognizing two right-angled triangles, we obtain

$$b^2 = c^2 + R^2 + 2cx.$$

Inserting $\mathrm{d}M = \frac{M}{2R}\mathrm{d}x$ and $b(x)$ into (11) we are ready to integrate

$$F_c = \frac{GmM}{2R}\int_{-R}^{R}\frac{c+x}{(c^2+R^2+2cx)^{3/2}}\,\mathrm{d}x.$$

Interpreting numerator and denominator as $u = c + x$ and $v' = (c^2+R^2+2cx)^{-3/2}$, we obtain on integration by parts of $\int uv'\mathrm{d}x$

$$F_c = \frac{GmM}{2R^2}\left(\frac{c+R}{\sqrt{(c+R)^2}} + \frac{c-R}{\sqrt{(c-R)^2}}\right).$$

On closer inspection of the terms in brackets we marvel at the emergence of a discontinuous function

$$F_c = \frac{GmM}{2R^2}\big(\mathrm{sign}(c+R) + \mathrm{sign}(c-R)\big).$$

This is exactly Newton's shell theorem. If $c < R$, so that m is inside the shell, then $|F_c| = 0$. If $c > R$, so that m is outside the shell, then $|F_c| = GMm/R^2$. It goes without saying that $c, R \in \mathbb{R}$ and $c, R > 0$ on physical grounds.

The law of gravitation (10) is inversely proportional to the squared radius and the effective mass direct proportional to its third power. Consequently, the gravitational force on a mass inside earth decreases linearly with the radius. With earth's surface as datum ($g = 9.81\,\mathrm{m/s^2}$) we obtain for the gravitational force

$$F_g = \frac{r}{R}mg$$

and the corresponding potential energy with zero level at $r = R$

$$E_{\mathrm{pot}} = \frac{1}{2}\frac{mg}{R}(r^2 - R^2).$$

Conservation of energy with kinetic energy of a point mass, at rest at $r = R$, leads to the position-dependent velocity (3).

A.2 A Special Case of Noether's Theorem

Noether's theorem is very powerful and you may become motivated to learn more about it by the benefit we gain in our case with one scalar independent variable and its derivative. Indeed, we merely demonstrate its correctness and do not go into its motivation by concepts of symmetry.

Consider a set of generalized coordinates and velocities x and x', respectively. For a Lagrangian without explicit dependence on time (generalizable to functionals that are invariant under time-translations of start and end time)

$$S = \int_{t_0}^{t_1} L\big(x(t), x'(t)\big)\,\mathrm{d}t,$$

the necessary condition for an extremum takes the form

$$\delta S = 0 = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial x'}\delta x'\right)\mathrm{d}t.$$

By invoking arbitrariness of the variations δx everywhere but at the integration boundaries t_0, t_1 where we demand $\delta x = 0$, partial integration leads to the celebrated *Euler-Lagrange equation*

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x'} - \frac{\partial L}{\partial x}. \tag{12}$$

Now, multiplying equation (12) by x' and adding a seemingly useless zero $0 = x''\frac{\partial L}{\partial x'} - x''\frac{\partial L}{\partial x'}$ helps to recognize a precious conserved quantity

$$\begin{aligned}
0 &= x'\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x'} - \frac{\partial L}{\partial x}\right),\\
0 &= x''\frac{\partial L}{\partial x'} + x'\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x'} - x'\frac{\partial L}{\partial x} - x''\frac{\partial L}{\partial x'},\\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(x'\frac{\partial L}{\partial x'} - L\right).
\end{aligned}$$

As example, consider a particle in a homogeneous gravitational field with Lagrangian (x being the vertical coordinate upwards)

$$L = E_{\mathrm{kin}} - E_{\mathrm{pot}} = \frac{1}{2}mx'^2 - mgx.$$

The Lagrangian as the difference between kinetic and potential energy may appear somewhat obscure. On the other hand the conserved quantity, refered to as Hamiltonian

$$H = x' \frac{\partial L}{\partial x'} - L = \frac{1}{2} m x'^2 + m g x$$

corresponds in this example to the total mechanical energy, which is more obviously meaningful for the analysis of the particle motion.

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